MATHEMATICS PROBLEM 156

How many integral right triangles are there with longest side less than or equal to *100* ?

Each person submitting a correct solution with a correct explanation to Problem 156 by noon Friday 14 April 2006 will be recognized as a solver on the next problem. Submit solutions to Prof. Wardlaw at mathprob@usna.edu (please no attachments!).

Correct solutions to Math Problem 155 were submitted by Midshipman Daniel Ryan and James Santelli, LT Josh Wood, and by Professors Sommer Gentry, Russell Jackson, and Mark Meyerson. Midshipmen Christopher Landis and Reeve Meck submitted the correct number without justification. My solution to Math Problem 155 is on the back of this page and on the Math Dept bulletin board on the third floor of Chauvenet Hall across from the Mathematics Department Office.

MATHEMATICS PROBLEM 155

We can represent a triangle with sides of length a, b, c by the ordered triple (a, b, c). Changing the order of the sides doesn't change the triangle, so $(a, b, c), (b, a, c), (b, c, a), (c, b, a), (c, a, b)$, and (a, c, b) all represent the same triangle. To avoid confusion, let's agree to write (a, b, c) with $a \leq b \leq c$. We say that a triangle (a, b, c) is *integral* if $a, b,$ and c are integers.

How many integral triangles are there with longest side less than or equal to *100* ?

Solution. There are 87,125 integral triangles with longest side less than or equal to *100* . To see this, we can count the ordered triples (a, b, c) with $1 \leq a \leq b \leq c \leq 100$ which satisfy the triangle inequality $c < a + b$. To get some ideas, let's start small:

$c = 1$:	$(1, 1, 1)$		1
$c = 2$:	$(1, 2, 2)$		2
	$(2, 2, 2)$		
$c = 3$:	$(2, 2, 3)$	$(1, 3, 3)$	$1 + 3$
		$(2, 3, 3)$	

		$(3, 3, 3)$	
$c = 4$:	$(2, 3, 4)$	$(1, 4, 4)$	$2 + 4$
	$(3, 3, 4)$	$(2, 4, 4)$	
		$(3, 4, 4)$	
		$(4, 4, 4)$	

Going on for a few more values of c, we observe that when $c = 2k - 1$ is odd, the inequality $c < a + b \leq 2b$ implies that the smallest possible b is $b = k$, and there is only one triangle $(k, k, 2k - 1)$ with $b = k$ and $c = 2k-1$. If $k < 2k - 1$, there are three triangles $(k-1, k+1, 2k-1)$, $(k, k+1, 2k-1)$, and $(k+1, k+1, 2k-1)$. In fact, for each of the k values $b = k, k+1, \ldots, 2k-1$, a can assume any of the $2(b-k)+1$ values satisfying $2k-b \leq a \leq b$. Thus when $b = k + j$, there are $2j + 1$ triangles of the form $(a, k+j, 2k-1)$, and the total number of triangles when $c = 2k-1$ is the sum $1 + 3 + \ldots + c$ of the odd positive integers from 1 to c.

When $c = 2k$ is even, the inequality $c < a + b \leq 2b$ implies that the smallest possible b is $b = k+1$, and there are two triangles $(k, k+1, 2k - 1)$ and $(k+1, k+1, 2k - 1)$ with $b = k+1$ and $c = 2k$. In fact, for each of the k values $b = k+1, k+2, \ldots, 2k$, a can assume any of the $2(b-k)$ values satisfying $2k-b < a \leq b$. Thus when $b = k + j$, there are $2j$ triangles of the form $(a, k+j, 2k)$, and the total number of triangles when $c = 2k$ is the sum $2 + 4 + \ldots + c$ of the even positive integers from 1 to c.

Putting these together, for each $k = 1, 2, \ldots, 50$, there are $s(2k) = 1 + 2 + \ldots + 2k = k(2k + 1)$ triangles with longest side $c = 2k - 1$ or $c = 2k$. Thus the number of integral triangles with longest side less than or equal to 100 is $T(100) = Sum(s(2k), k = 1..50) = Sum(2k^2 + k, k = 1..50) = 87,125$, as can be found using the Voyage 200 calculator or Maple. One can also find them "by hand" using the formulas $s(n) = 1 + 2 + \ldots + n = n(n+1)/2$ and $q(n) = 1^2 + 2^2 + \ldots + n^2 = n(n + 1)(2n + 1)/6$. (See Thm 3 Appendix E on page A37 of *Calculus - Early Transcendentals* 5e by James Stewart. These formulas are easily proved by induction on n.)