MATHEMATICS PROBLEM 157

Consider the following game: You roll a pair of dice again and again until you get a seven. If you first get a seven on the *n* th roll, you are paid n^2 dollars.

What would be a fair price to pay in order to play this game? Or, more accurately, what is the expected value of the game?

Each person submitting a correct solution with a correct explanation to Problem 157 by noon Friday 5 May 2006 will be recognized as a solver on the next problem. Submit solutions to Prof. Wardlaw at mathprob@usna.edu (please no attachments!).

Correct solutions to Math Problem 156 were submitted by Midshipmen Christopher Landis and George Schaertl, LT Josh Wood, and by Professors Sommer Gentry, Russell Jackson, and Mark Meyerson. My solution to Math Problem 156 is on the back of this page and on the Math Dept bulletin board on the third floor of Chauvenet Hall across from the Mathematics Department Office.

MATHEMATICS PROBLEM 156

How many integral right triangles are there with longest side less than or equal to *100* ?

Solution. There are 52 right triangles with longest side less than or equal to 100. To see this we must count the Pythagorean triples (A, B, C) with positive integers A, B, and $C \leq 100$ which satisfy $A^2 + B^2 = C^2$. First, observe that if $A = 2j + 1$ and $B = 2k + 1$ are both odd, then $A^2 + B^2 = 4j^2 + 4j + 1 + 4k^2 + 4k + 1 = 4m + 2$ cannot be equal to C^2 since any even square must be divisible by 4.

Assume (A, B, C) is a Pythagorean triple (PT). Then at least one of A and B must be even. Let gcd(A, B, C) = d and a = A/d, b= B/d, and c = C/d. Then (a, b, c) is a primitive Pythagorean triple (PPT), that is, it is a PT and gcd(a, b, c) = 1, so its entries have no factors in common. Moreover, we can assume that b is even and a and c are both odd. Hence $b^2/4 = (c^2 - a^2)/4 = [(c - a)/2][(c + a)/2] = xy$ with $x = (c - a)/2$ and $y = (c + a)/2$ both positive integers. Since $y + x = c$ and $y - x = a$ have no factors

in common, neither can x and y; that is, $\gcd(x, y) = 1$. But $xy = b^2/4$ is a square, so both x and y must be squares. Thus there are positive integers u and v such that $y = u^2$ and $x = v^2$. It follows that $a = u^2 - v^2$, $b = 2uv$, and $c = u^2 + v^2$. Clearly, $0 < v < u$ and u and v have no factors in common. Since a and c are both odd, it follows that u and v have opposite parities, that is, one is even and the other is odd. Moreover, if u and v are integers satisfying $0 < v < u$, $\gcd(u, v) = 1$, and having opposite parities, then

$a = u^2 - v^2$, $b = 2uv$, and $c = u^2 + v^2$ implies that (a, b, c) is a PPT.

Now we see we can count every PT $(A, B, C) = (da, db, dc)$ with $C \leq 100$ by counting all of the PPTs and their appropriate multiples. We can make sure we get all of the PPTs by listing the appropriate (u, v) pairs. This is done in the following table:

u	v	(a, b, c)	# of multiples (A, B, C)
2	1	(3, 4, 5)	20
3	2	(5, 12, 13)	7
4	1	(15, 8, 17)	5
4	3	(7, 24, 25)	4
5	2	(21, 20, 29)	3
5	4	(9, 40, 41)	2
6	1	(35, 12, 37)	2
6	5	(11, 60, 61)	1
7	2	(45, 28, 53)	1
7	4	(33, 56, 65)	1
7	6	(13, 84, 85)	1
8	1	(63, 16, 65)	1
8	3	(55, 48, 73)	1
8	5	(39, 80, 89)	1
8	7	(15, 112, 113)	0
9	2	(77, 36, 85)	1
9	4	(65, 72, 97)	1
		Total:	52