MATHEMATICS PROBLEM 158

To check whether a positive integer is divisible by 7, subtract twice the last digit from the number left after removing the last digit of the original number. The original number is divisible by 7 if and only if the resulting number is divisible by 7. For example, 1652 is divisible by 7 because 165 - 2x2 = 161 is, and 161 is divisible by 7 since 16 - 2x1 = 14 is. 1776 is not divisible by 7 since 177 - 2x6 = 165 is not divisible by 7 since 16 - 2x5 = 6 is not.

Explain (with a proof) why this method works.

Each person submitting a correct solution with a correct explanation to Problem 158 by noon Friday 20 October 2006 will be recognized as a solver on the next problem. Submit solutions to Prof. Wardlaw at mathprob@usna.edu.

Correct solutions to Math Problem 157 were submitted by Professors Mark Kidwell and Mark Meyerson. My solution to Math Problem 157 is on the back of this page and on the Math Dept bulletin board on the third floor of Chauvenet Hall across from the Mathematics Department Office.

MATHEMATICS PROBLEM 157

Consider the following game: You roll a pair of dice again and again until you get a seven. If you first get a seven on the *n* th roll, you are paid n^2 dollars.

What would be a fair price to pay in order to play this game? Or, more accurately, what is the expected value of the game?

Solution. Six of the 36 equally likely outcomes when rolling a pair of dice give a sum of 7, namely 1 + 6 = 2 + 5 = 3 + 4 = 4 + 3 = 5 + 2 = 6 + 1. Thus, the probability of getting a seven on a single throw is p = 1/6, and the probability of not getting a seven is q = 5/6. In order to first get a seven on the nth roll, you must first roll n - 1 non-sevens

and then roll a seven, so the probability of doing this is $P(n) = (q^{(n-1)})p$. The expected value of the game is the infinite sum

$$EV = \text{Sum}((n^2)P(n), n = 1,2, \ldots) = \text{Sum}((n^2)(p)(q^{(n-1)}), n = 1, 2, \ldots)$$

$$= (p)\text{Sum}((n^2)(q^{(n-1)}), n = 1, 2, \ldots).$$

To evaluate the latter infinite sum, note that the infinite geometric sum is

$$1/(1 - x) = \text{Sum}(x^n, \quad n = 0, 1, 2, \ldots) \quad \text{when } |x| < 1.$$

Differentiating both sides of the latter gives

$$1/(1 - x)^2 = \text{Sum}(nx^{(n - 1)}, n = 1, 2, \ldots).$$

Multiplying both sides of this by x then gives

$$x/(1 - x)^2 = \text{Sum}(nx^n, \ n = 1, 2, \ldots).$$

Differentiating again gives

$$(1+x)/(1 - x)^3 = \text{Sum}((n^2)x^{(n - 1)}, \ n = 1, 2, \ldots).$$

Now taking $x = q = 5/6$ and $1 - x = 1 - q = p = 1/6$ in the above expression for the expected value gives

$$EV = (1 - x)\,\text{Sum}((n^2)x^{(n - 1)}, \ n = 1, 2, \ldots) = (1 + x)/(1 - x)^2$$

$$= (1 + 5/6)/(1 - 5/6)^2 = (11/6)/(1/6)^2 = 66.$$

Hence, \$66 would be a fair price to pay to play this game.