MATHEMATICS PROBLEM 159

Let A be the 3 x 3 matrix over the integers modulo 3 with first row $A1 = [0\ \ 1\ \ 0]$, second row $A2 = [0\ \ 0\ \ 1]$, and third row $A3 = [2\ \ 1\ \ 0]$.

a. Show that A has multiplicative order 26. That is, $A^{26} = I$ is the identity matrix, but A^k is not I for any positive integer $k < 26$. (All calculations are done modulo 3.)

b. Show that the sum of any two powers of A is either a power of A or the zero matrix. (Again, all calculations are done modulo 3.)

Each person submitting a correct solution with a correct explanation to Problem 159 by noon Friday 10 November 2006 will be recognized as a solver on the next problem. Submit solutions to Prof. Wardlaw at wpw@usna.edu.

Correct solutions to Math Problem 158 were submitted by Midshipmen Sam Reno and Jared Wolterstorff, LT Josh Wood, and Professors Russell Jackson (who also should have been recognized for his solution to MP157), Amy Ksir, and Mark Meyerson. My solution to Math Problem 157 is on the back of this page and on the Math Dept bulletin board on the third floor of Chauvenet Hall across from the Mathematics Department Office.

MATHEMATICS PROBLEM 158

To check whether a positive integer is divisible by 7, subtract twice the last digit from the number left after removing the last digit of the original number. The original number is divisible by 7 if and only if the resulting number is divisible by 7. For example, 1652 is divisible by 7 because 165 - 2x2 = 161 is, and 161 is divisible by 7 since 16 - 2x1 = 14 is. 1776 is not divisible by 7 since 177 - 2x6 = 165 is not divisible by 7 since 16 - 2x5 = 6 is not.

Explain (with a proof) why this method works.

Solution. Any positive integer n can be written as $n = 10t + u$, where u is a single digit number. Then $n = 10t + u$ is divisible by 7 if and only if $n = 10t + u - 21u$ is divisible by 7, since $21u$ is divisible by 7. Now $10t + u - 21u = 10t - 20u = 10(t - 2u)$ is divisible by 7 if and only if $t - 2u$ is divisible by 7, since 10 has no factors in common with 7. This completes the proof!

Notes. $n = 10t + u$ is divisible by 3 if and only if $t - 2u$ is divisible by 3 because 3 divides 21 but doesn't divide 10, so same trick works for 3 as well as 7. Of course, we know n is divisible by 3 if and only if the sum of its digits is divisible by 3.