MATHEMATICS PROBLEM 161

The *residue* of an integer n modulo an integer $d > 1$ is the remainder r left when n is divided by d. That is, if $n = dq + r$ for integers q and r with $0 \leq r < d$, we write $r = n \pmod d$ for the residue of n modulo d.

Show that the residue modulo 7 of a (large) integer n can be found by separating the integer into 3-digit blocks

$$n = b(s)b(s-1)\ldots b(1)$$

(Note that b(s) may have 1, 2, or 3 digits, but every other block must have exactly three digits.) Then the residue modulo 7 of n is the same as the residue modulo 7 of $b(1) - b(2) + b(3) - b(4) + \ldots \pm b(s)$. For example,

$$n = 25{,}379{,}885{,}124{,}961{,}154{,}398{,}521{,}655 \pmod 7$$
$$= 655 - 521 + 398 - 154 + 961 - 124 + 885 - 379 + 25 \pmod 7$$
$$= 1746 \pmod 7 = 746 - 1 \pmod 7 = 745 \pmod 7 = 3 \pmod 7.$$

Explain why this works and show that the same stunt works for residues modulo 13.

Each person submitting a correct solution with a correct explanation to Mathematics Problem 161 by noon Friday 26 January 2007 will be recognized as a solver when the next problem is announced. Submit solutions to Prof. Wardlaw at wpw@usna.edu.

Correct solutions to Mathematics Problem 160 were submitted by Midshipmen Samuel Reno and Erik Rye, Mr. Daniel James of Auburn, AL, and Professor Mark Meyerson. My solution to Mathematics Problem 160 is on the back of this page and on the Mathematics Department bulletin board on the third floor of Chauvenet Hall across from the Mathematics Department Office.

MATHEMATICS PROBLEM 160

This problem continues the ideas of Mathematics Problem 158. To check whether a positive integer is divisible by 13, add four times the last digit to the number left after removing the last digit of the original number. The original number is divisible by 13 if and only if the resulting number is divisible by 13. That is, $n = 10t + u$ is divisible by 13 if and only if $t + 4u$ is. For example, 3094 is divisible by 13 because $309 + 4x4 = 325$ is, and 325 is divisible by 13 since $32 + 4x5 = 52$ is. 1776 is not divisible by 13 since $177 + 4x6 = 201$ is not divisible by 13 since $20 + 4x1 = 24$ is not.

Explain (with a proof) why this method works. Then produce similar rules for divisibility by 17 and by 19, and explain why they work.

Solution. a. **$n = 10t + u$ is divisible by 13** iffi $10t + u + 39u = 10(t + 4u)$ is divisible by 13 iffi **$m = t + 4u$ is divisible by 13**, since 10 has no factors in common with 13.

b. **$n = 10t + u$ is divisible by 17** iffi $10t + u - 51u = 10(t - 5u)$ is divisible by 17 iffi **$m = t - 5u$ is divisible by 17**, since 10 has no factors in common with 17.

c. **$n = 10t + u$ is divisible by 19** iffi $10t + u + 19u = 10(t + 2u)$ is divisible by 19 iffi **$m = t + 2u$ is divisible by 19**, since 10 has no factors in common with 19.