MATHEMATICS PROBLEM 162

Plebes may be interested in attending the Math Open House on Tuesday 20 February 2007 on the first deck of Chauvenet Hall to find out about the Mathematics Major. There will be a Sudoku room.

See if you can discover the secret three word message in two consecutive rows of the completion of the Sudoku below. (The Sudoku was kindly supplied by Prof. Meyerson.)

Sudoku rules: Fill in the blanks so only 9 different letters appear, and every row and column contains 9 different letters. Also, if the 9x9 puzzle is divided into 9 blocks of size 3x3, each of these 3x3 blocks must contain 9 different letters. The solution is unique.

	R			T	I	N	M	A
M	J			R		T	O	I
I	A		O			H	J	
	I		M	N		O	H	T
N	O	R	I		T	J	A	M
	T						I	
J		O	R	I			T	
	M		T			I	N	J
T	H		N		J	M	R	O

Each person submitting a correctly completed Sudoku and three word message to Mathematics Problem 162 by noon Friday 16 February 2007 will be recognized as a solver when the next problem is announced. Submit solutions to Prof. Wardlaw at wpw@usna.edu.

Correct solutions to Mathematics Problem 161 were submitted by Professors Russell Jackson and Mark Meyerson. My solution to Mathematics Problem 161 is on the back of this page and on the Mathematics Department bulletin board on the third floor of Chauvenet Hall across from the Mathematics Department Office.

MATHEMATICS PROBLEM 161

The *residue* of an integer n modulo an integer $d > 1$ is the remainder r left when n is divided by d. That is, if $n = dq + r$ for integers q and r with $0 \leq r < d$, we write $r = n \pmod{d}$ for the residue of n modulo d.

Show that the residue modulo 7 of a (large) integer n can be found by separating the integer into 3-digit blocks

$$n = b(s)b(s\text{-}1)\ldots b(1)$$

(Note that b(s) may have 1, 2, or 3 digits, but every other block must have exactly three digits.) Then the residue modulo 7 of n is the same as the residue modulo 7 of b(1) - b(2) + b(3) - b(4) + … ± b(s). For example,

$$n = 25{,}379{,}885{,}124{,}961{,}154{,}398{,}521{,}655 \pmod 7$$
$$= 655 - 521 + 398 - 154 + 961 - 124 + 885 - 379 + 25 \pmod 7$$
$$= 1746 \pmod 7 = 746 - 1 \pmod 7 = 745 \pmod 7 = 3 \pmod 7.$$

Explain why this works and show that the same stunt works for residues modulo 13.

Solution

$$n = b(s)b(r\text{-}1)\ldots b(1) = \text{sum}(b(k+1)1000\text{^}k,\ k = 0..s\text{-}1)$$

$$= \text{sum}(b(k+1)(-1)\text{^}k,\ k = 0..s\text{-}1) = b(1) - b(2) + \ldots + (-1)b(s) \pmod 7$$

because 1001 = 7x11x13 implies that 1000 = -1 (mod 7) and 1000^k can be replaced by (-1)^k in the sum. Similarly, 1000 = -1 (mod 13), so same trick works mod 13.

Professor Mark Meyerson's solution is a bit easier to understand without a knowledge of modular arithmetic:

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We have n=sum(b(k+1)1000^k,k=0..s-1).  Let d=n-[b(1)-b(2)+...+-b(s)]
=sum(b(k+1)(1000^k-(-1)^k),k=0..s-1).  But a^k-b^k=(a-b)(a^(k-1)+a^(k-2)
b+...+b^(k-1)), so 1001=1000-(-1) divides d since it divides each term
in the sum.  Since 1001=7x11x13, the result holds for 7 and 13 (also 11,
77, etc.).
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