MATHEMATICS PROBLEM 164

Two lines are fixed in three space at a distance d apart and have an angle θ between them. Place a line segment of length a on one of these lines and another line segment of length b on the other line. Find the volume of the convex hull of these two line segments.

Note: A set S of points is convex if given any two points in the set, the line segment joining these two points is also in the set S. The convex hull of a set T is the smallest convex set containing the set T.

Each person submitting a correct solution to Mathematics Problem 164 by 1200 Thursday 5 April 2007 will be recognized as a solver when the next problem is announced. Submit solutions to Prof. Wardlaw at wpw@usna.edu.

Correct solutions to Mathematics Problem 163 were submitted by Midshipmen Kevin McCadden, Alexander McIntosh (the latter without justification), and Karl Motoyama, and Professors Russell Jackson and Mark Meyerson. Professor Jackson's solution to Mathematics Problem 163 is on the back of this page and on the Mathematics Department bulletin board on the third floor of Chauvenet Hall across from the Mathematics Department Office.

MATHEMATICS PROBLEM 163

Find a function f such that the inverse function f^{-1} of f is the same as the derivative f' of f. That is, $f^{-1} = f'$. (Recall that the inverse function is defined by $f^{-1}(y) = x$ if and only if $f(x) = y$.)

(This problem was proposed by Professor Irina Popovici.)

<u>Solution by Prof. Russell Jackson:</u>

Suppose that $f'(x) = f^{-1}(x)$.

Applying f to both sides yields $x = f(f'(x))$.

We look for solutions f that have the form

$$f(x) = ax^n.$$

Applying this ansatz, we have $x = f(f'(x)) = a\,(nax^{n-1})^n$ and so we must find a and n to solve the equation

$$x = a^{n+1}n^n x^{n(n-1)}.$$

Matching the exponent of x on the left and the right, we have $1 = n(n-1)$, which has positive root $n = (1+\sqrt{5})/2$, that is, the golden ratio, φ.

$$n = \varphi \ \ (=(1+\sqrt{5})/2).$$

Matching the coefficient of x, we have $1 = a^{\varphi+1}\varphi^{\varphi}$. And so $a = \varphi^{-\varphi/(\varphi+1)}$. Now since $\varphi+1 = \varphi^2$ we can simplify, writing $a = \varphi^{-1/\varphi}$. And since we also have $1/\varphi = \varphi-1$, $a = \varphi^{-1/\varphi} = \varphi^{1-\varphi}$.

$$a = \varphi^{1-\varphi}.$$

And so our function is $f(x) = \varphi^{1-\varphi}x^{\varphi} = \varphi(x/\varphi)^{\varphi}$.

$$\mathbf{f(x) = \varphi(x/\varphi)^{\varphi},} \text{ where } \varphi =(1+\sqrt{5})/2.$$