

Image Compression using Matrices

by Asst. Prof. Vrej Zarikian (zarikian@usna.edu)

Most students first encounter matrices in algebra. They learn that matrices multiply in a very peculiar fashion (row-by-column):

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 4 \\ 11 & 3 & 8 \\ 17 & 5 & 12 \\ 23 & 7 & 16 \end{bmatrix} = C.$$

Because multiplying matrices together is hard, so is factoring them apart. Nonetheless, every matrix can be factored efficiently on a computer into a product of three special matrices (this is called the *singular value decomposition*):

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 5 \end{bmatrix} = USV = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$

Notice that S is a *diagonal* matrix with decreasing diagonal entries and that U and V are *orthogonal* matrices (their columns are perpendicular unit vectors). Now suppose we change the smallest diagonal entry of S from 2 to 0, producing a new diagonal matrix S' . Then

$$US'V = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = A'.$$

Notice that A' is not substantially different from A (no entry is off by more than 1). Also, since the 3rd row and the 3rd column of S' are entirely 0,

we can obtain A' by multiplying just the first two columns of U , just the upper 2-by-2 block of S' , and just the first two rows of V :

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = A'.$$

These observations have important practical consequences. Let's say you have a big digital image (1000 × 1000 pixels) that you want to send over a slow internet connection. As far as the computer is concerned, the image is a 1000 × 1000 matrix A , where each entry of the matrix is a number representing the color of the matching pixel. You can compute the singular value decomposition of A :

$$A = [1000 \times 1000] = USV = [1000 \times 1000] [1000 \times 1000] [1000 \times 1000].$$

Then you can “zero out” a bunch of the diagonal entries of S , say the smallest 900, producing new diagonal matrix S' . The matrix $A' = US'V$ is a (hopefully good) approximation to the matrix $A = USV$, but it can be computed by multiplying just the first 100 columns of U , just the upper 100-by-100 block of S' , and just the first 100 rows of V :

$$A \approx A' = [1000 \times 100] [100 \times 100] [100 \times 1000].$$

Instead of transmitting A (1,000,000 entries), you transmit just the first 100 columns of U (100,000 entries), just the upper 100-by-100 block of S' (10,000 entries), and just the first 100 rows of V (100,000 entries). That's a 79% reduction in the information sent. The computer at the other end multiplies the matrices back together to get A' , which should look almost like A . Sounds good, but does it work? You be the judge—which of the following two images is the original, and which has been compressed?



Mids who are interested in learning more about matrices should take SM261 (Matrix Theory) which is offered mainly in the Fall.

*On the cover: The USNA seal (left) and the Mandelbrot set (right).
 Editors: Assoc. Prof. T. S. Michael (tsm@usna.edu) and Asst. Prof. Vrej Zarikian (zarikian@usna.edu).

USNA Profs, former Mid honored at Mathfest

Three USNA math professors and a former USNA midshipman were honored at the MAA Mathfest held in Portland, Oregon, August 5-8, 2009. MAA stands for the Mathematical Association of America, and Mathfest is its annual national meeting.



Asst. Prof. Sommer Gentry was one of only three professors in the country to receive the Henry L. Alder Award for Distinguished Teaching by a beginning college or university faculty member. For their article “Enumerative Algebraic Geometry of Conics” which appeared in the *American Mathematical Monthly*, Assoc. Prof. Amy Ksir, Prof. Will Traves, and Trident Scholar Andrew Bashelor ('05) received one of five Lester R. Ford Awards for expository writing. The same article was also awarded the (only) Merten M. Hasse Prize for a noteworthy paper appearing in an MAA publication.

Faculty Profile: Capt. Rob Lantz, USMC

Capt. Rob Lantz, USMC, was raised in Petersburg, West Virginia. Following high school he accepted an appointment to the US Naval Academy and started Plebe Summer with the Class of 2000. He graduated in May 2000 with a Marine commission and a BS in Mechanical Engineering. Following The Basic School he was awarded the Logistics Officer MOS and accepted orders to Camp Lejeune for duty as a platoon commander in the 2nd Transportation Support Battalion. During the next three years he held various billets during diverse deployments. It was at this time that he gained an appreciation for the inherently quantitative nature of the operational and strategic problems of interest to the Marine Corps.

In May 2004 Capt. Lantz accepted orders to the Naval Postgraduate School, where he obtained an MS in Operations Research (OR).

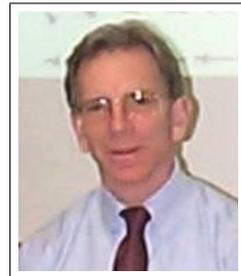


At NPS he learned the theory and methods associated with the solution of the quantitative problems that he had personally encountered. He became familiar with techniques such as optimization, data analysis and simulation modeling, as well as the technology used to implement them. He completed a thesis titled *A Data Mining Approach to Forecasting IED Placement in Space and Time*. To this day, it is the application of math to relevant and timely military problems that fuels Captain Lantz’s passion for OR. In October 2006 Capt. Lantz began his tour as a Math Instructor at USNA. He has taught calculus, probability, and statistics, as well as a spreadsheet modeling class he helped develop. He is especially passionate about giving midshipmen the quantitative skills to succeed as junior officers in the Fleet Navy and Marine Corps. Capt. Lantz is married with two sons. Jackson is two-and-a-half and Connor is only two weeks old (born on November 10th, the birthday of the Marine Corps). Capt. Lantz’s interests include outdoor sports, marksmanship and apologetics.

From the Chair

by Prof. Geoff Price (glp@usna.edu)

As chair of the Math Department I welcome you to our side of the Naval Academy. I hope that you are learning a lot in your first course with us.



Your success in math will likely lead to success in all of your STEM classes. We have a dedicated staff of instructors who will help to guide you through our program. If you have difficulties with your course you are welcome to take advantage of the help we offer outside of the classroom. In Mathlab (Chauvenet 130) you will find a math instructor who can help you with your questions Monday through Friday during periods 1-6. And MGSP puts you in touch with upper-class midshipmen who can help you in the evening. Some of you may wish to consider a major in mathematics. With a diverse group of instructors we offer a wide range of courses which may appeal to you. Math is the gateway for many different types of careers. Come see us when we have our Open House during the spring semester. In the meantime you are welcome to see me (in Chauvenet 300) or your instructor to talk about your interest in mathematics.