SO335: Examples of the Material Derivative (Chapter 4)

1. Find the material rate of change of the temperature field at the point (50 m, 50 m) if the flow $\vec{u}$ is $\vec{u} = 5 \text{ m s}^{-1}\hat{i}$ and temperature is given by $T = 10°C + 0.2°C\left(e^{-\alpha t}\beta\left(x^2+y^2\right)\right)$.

<u>Solution</u>:

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (\vec{u}\cdot\nabla)T$$

$$\frac{DT}{Dt} = \frac{\partial}{\partial t}\left(10°C + 0.2°C\left(e^{-\alpha t}\beta\left(x^2+y^2\right)\right)\right) + \left[\left(5ms^{-1}\hat{i}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\right]\left(10°C + 0.2°C\left(e^{-\alpha t}\beta\left(x^2+y^2\right)\right)\right)$$

$$\frac{DT}{Dt} = \left(0.2°C\beta\left(x^2+y^2\right) - \alpha e^{-\alpha t}\right) + 5ms^{-1}\frac{\partial}{\partial x}\left(10°C + 0.2°C\left(e^{-\alpha t}\beta\left(x^2+y^2\right)\right)\right)$$

$$\frac{DT}{Dt} = \left(0.2°C\beta\left(x^2+y^2\right)\left(-\alpha e^{-\alpha t}\right)\right) + 5ms^{-1}\left(0.2°Ce^{-\alpha t}\beta\left(2x\right)\right)$$

Now solve at the point (50,50), and note that $\alpha$=1s$^{-1}$:

$$\frac{DT}{Dt} = 0.2°C\left(\frac{1}{2500m^2}\right)\left((50m)^2 + (50m)^2\right)\left(-1s^{-1}e^{-1s^{-1}t}\right) + 5ms^{-1}0.2°Ce^{-1s^{-1}t}\left(\frac{1}{2500m^2}\right)(2)(50m)$$

$$\frac{DT}{Dt} = e^{-ts^{-1}}\left(-0.4°Cs^{-1} + 0.04°Cs^{-1}\right)$$

*Colors (**red** and **black**) have been provided to help show the progression of each term.

2. Find the material acceleration if $\vec{u} = \alpha \sin(\beta x t)\hat{i} - \alpha \sin(\beta y t)\hat{j}$. $\alpha$ and $\beta$ are dimensional constants with units $ms^{-1}$ and $m^{-1}s^{-1}$, respectively.

Solution:

Note that for that vector, the components are $u = \alpha \sin(\beta x t)$ and $v = -\alpha \sin(\beta y t)$.

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u}$$

$$\frac{D\vec{u}}{Dt} = \frac{\partial}{\partial t}(\vec{u}) + \left[\left(u\hat{i} + v\hat{j} + 0\hat{k}\right) \cdot \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\right]\vec{u}$$

$$\frac{D\vec{u}}{Dt} = \frac{\partial}{\partial t}\left(\alpha \sin(\beta x t)\hat{i} - \alpha \sin(\beta y t)\hat{j}\right) + \alpha \sin(\beta x t)\frac{\partial}{\partial x}\left(\alpha \sin(\beta x t)\hat{i} - \alpha \sin(\beta y t)\hat{j}\right) - \alpha \sin(\beta y t)\frac{\partial}{\partial y}\left(\alpha \sin(\beta x t)\hat{i} - \alpha \sin(\beta y t)\hat{j}\right)$$

$$\frac{D\vec{u}}{Dt} = \alpha\beta x \cos(\beta x t)\hat{i} - \alpha\beta y \cos(\beta y t)\hat{j} + \alpha \sin(\beta x t)\left(\alpha\beta t \cos(\beta x t)\hat{i} - 0\hat{j}\right) - \alpha \sin(\beta y t)\left(-\alpha\beta t \cos(\beta y t)\hat{j}\right)$$

$$\frac{D\vec{u}}{Dt} = \hat{i}\left(\alpha\beta x \cos(\beta x t) + \alpha \sin(\beta x t)\alpha\beta t \cos(\beta x t)\right) + \hat{j}\left(-\alpha\beta y \cos(\beta y t) + \alpha \sin(\beta y t)\left(\alpha\beta t \cos(\beta y t)\right)\right)$$

$$\frac{D\vec{u}}{Dt} = \hat{i}\alpha\beta \cos(\beta x t)\left(x + \alpha t \sin(\beta x t)\right) + \hat{j}\alpha\beta \cos(\beta y t)\left(-y + \alpha t \sin(\beta y t)\right)$$

*Colors (**red**, **blue**, **magenta**, and **black**) have been provided to help show the progression of each term.