Lesson 4

- Objectives
  - Fundamental Forces
    - Inertial Reference Frame
      - Gravitational Force
      - Pressure Gradient Force (PGF)
      - Friction (Viscosity) Force
    - Non-inertial Reference Frame
      - Centrifugal Force
      - Coriolis Force
  - Conservation of Mass
    - Continuity Equation

Equations for a Nonrotating Fluid

- The state of the atmosphere or ocean can be defined by 5 variables
  - u, v, w, P and T
    - Recall, using the EOS ρ can be found from P and T
- We therefore need 5 independent equations (with boundary conditions) to determine the evolution of a fluid
  - The Laws of motion (3 directions)
  - Conservation of Mass
  - The 1st Law of Thermodynamics

Fundamental Forces

- In lesson 1 we made an assumption - that the ***atmosphere and ocean were continuous in structure and properties*** at our scale of interest
  - This means they are Newtonian fluids and therefore can use N2L, $\vec{F} = m\vec{a}$, to describe acceleration
    - The approach is slightly different than we saw in Physics
      - In Physics we dealt with discrete objects of **constant mass**
      - In Fluids we deal with a continuous field of matter with **varying density**

Law of Motion (N2L)

- Inertial frame of reference
  - A non-accelerating coordinate system
  - Follow the same parcel around - **therefore** the time derivative is a **total derivative**
- N2L becomes:

$$\vec{F} = \rho \, \delta x \, \delta y \, \delta z \frac{D\vec{U}}{Dt}$$

  - where:

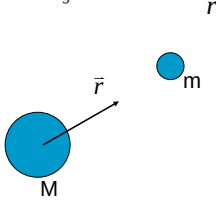
$$\frac{D\vec{U}}{Dt} = \frac{\partial \vec{U}}{\partial t} + (\vec{U} \bullet \nabla)\vec{U}$$

Gravitational Force (g*)

- Newton's Law of Universal Gravitation

Every particle in the universe attracts every other particle with a force that is proportional to the product of their masses and inversely proportional to the square distance between them. The force acts along the line joining the two particles.

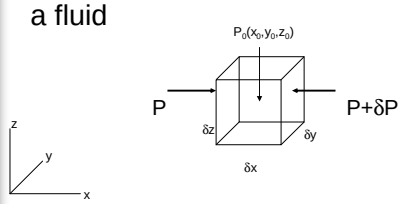
$$\vec{F}_g \equiv \vec{g}^* = -G\frac{Mm}{r^2}\vec{r}$$



$$\frac{\vec{F}_g}{m} \equiv \vec{g}^* = -G\frac{M}{a^2}\vec{r}$$

Pressure Gradient Force (PGF)

- P = F/A
- Force due to ***variation*** of pressure within a fluid



$$\frac{\vec{F}_p}{m} = -\frac{1}{\rho}\left(\frac{\partial P}{\partial x}\hat{i} + \frac{\partial P}{\partial y}\hat{j} + \frac{\partial P}{\partial z}\hat{k}\right) = -\frac{1}{\rho}\nabla P$$

Pressure Gradient Force (PGF)

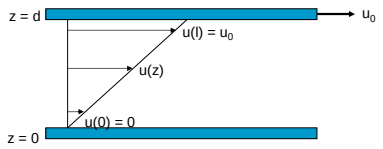
- Example
  - Consider a SLP of 1000.3 hPa from a station near the center of a low, with a temperature of 9 C. 500 km away a second station reports a SLP of 1000.9 hPa. What is the pressure gradient force?
    - Hint: IGL is needed

Frictional (Viscous) Force

- Molecular friction
  - Accelerates (±) a body that is in contact with another body moving at a different speed
  - Molecules collide with those of another
    - If one object moves (has momentum: p = mv), the other object will receive some of it and move
- Eddies can exchange momentum and therefore cause frictional forces
  - Wind experiences a retarding force: F_{fr} (ground)
  - The ground experiences an accelerating force
    - Free moving surface - waves, currents…

Viscous Force

- A fluid is a state that continuously yields to any shear stress
- Here a solid exerts a shear stress on a fluid



Viscous Force

- Force on upper plate $F = \frac{\mu A u_0}{d} = \mu A \frac{\partial u}{\partial z}$
  - μ ("mu"): dynamic (or shear) viscosity coefficient
    - Property of the fluid
- This force must just equal the force exerted by the upper plate on the fluid immediately below it (N3L)
- State of uniform motion
  - Every horizontal layer of fluid must exert the same force on the fluid below it

Viscous Force

- The shear stress - τ ("tau") - exerted on the fluid by the upper plate is proportional to the velocity of the plate and inversely proportional to the distance between the plates (d)
- The x-component of the shear stress can be written: $\tau \hat{i} = \mu \frac{\partial u}{\partial z}$
  - It is a *flux of momentum*
    - Momentum per unit time per unit area

Shear Stress

- Defined steady state flow - no acceleration and therefore no net viscous force
  - The shear stress (internal friction) does not cause a net force
- So, there must be a gradient in the shear stress

$$\frac{1}{\rho}\frac{\partial \tau}{\partial z} = \frac{\mu}{\rho}\frac{\partial}{\partial z}\left(\frac{\partial \mu}{\partial z}\right) = \upsilon \frac{\partial^2 u}{\partial z^2} \qquad \upsilon = \mu / \rho$$

Shear Stress

- υ ("nu") is the kinematic viscosity coefficient
  - Atmosphere: 1.46 x 10^{-5} m^2/s
  - Ocean: 1.0 x 10^{-6} m^2/s
- Shear stress is a **vector**
  - Shear stress has nine components (3D)

$$F_x = \upsilon \nabla^2 u \qquad \vec{F}_{shear}\hat{i} = \upsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$F_y = \upsilon \nabla^2 v$$

$$F_z = \upsilon \nabla^2 w$$

Apparent Forces

- Non-inertial reference frame
  - An accelerating reference frame
  - Uniform circular motion
    - Centrifugal force
    - Coriolis force

Centrifugal Acceleration

- Appears to deflect the moving body *outward* from center of curvature
  - Equal and opposite to centripetal acceleration
  - Always perpendicular to Earth's axis; directed outward

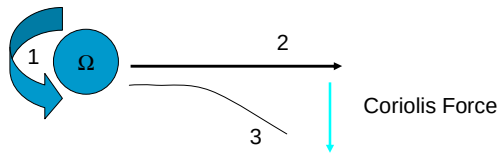
$$\left|\frac{\vec{F}_{cf}}{m}\right| = \Omega^2 \vec{R}$$

Effective Gravity

- Resultant of gravitational and centrifugal forces
- Directed perpendicular to local tangent of Earth, not toward the center

Coriolis Force

- The force that appears to act on a body moving relative to a rotating system, when viewed from that system
- Merry-Go-Round Example



Coriolis Acceleration

- The coriolis acceleration describes the tendency for fluid parcels to turn
- In the absence of other forces, a parcel would accelerate as:

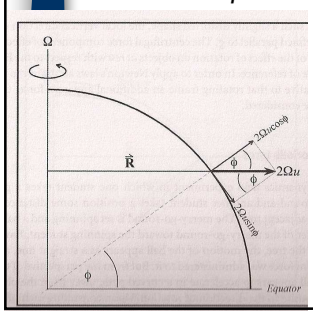
$$\frac{D\vec{U}}{Dt} = -2\Omega \times \vec{U}$$

- By definition: CF does no work, it only acts to change the direction of flow

Coriolis Acceleration: Derivation

- Recall the coriolis parameter:

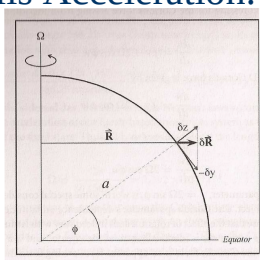
$$f = 2\Omega \sin\phi$$



$$\frac{dw}{dt} = 2\Omega u \cos\phi$$

$$\frac{dv}{dt} = -2\Omega u \sin\phi = -fu$$

Coriolis Acceleration: Derivation



- Horizontal $\frac{du}{dt} = 2\Omega \sin\phi (a \frac{d\phi}{dt}) = fv$
- Vertical $\frac{du}{dt} = -2\Omega \cos\phi (\frac{dz}{dt}) = -2\Omega \cos\phi w$

Coriolis Acceleration: Full 3-D

$$\frac{du}{dt} = fv - 2\Omega\cos\phi w$$

$$\frac{dv}{dt} = -fu$$

$$\frac{dw}{dt} = 2\Omega u\cos\phi$$

Coriolis Acceleration

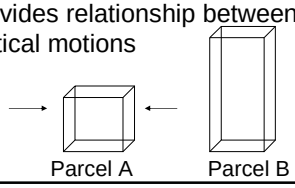
- Example
  - The Coriolis acceleration can be calculated using observations.
  - City X, has a latitude of 40N and reports a wind of 5 kts from the North.
  - Calculate the Coriolis acceleration

Basic Laws of Physics

- Conservation of:
  - Mass
    - Leads to Continuity Equation
    - SO335 and SO414
  - Momentum
    - Leads to EOM
    - SO414
  - Energy
    - Leads to 1st Law of Thermodynamics
    - SO345

Conservation of Mass

- Cannot create or destroy mass - 1 kg of a fluid will always be 1 kg of that fluid
  - If fluid is compressible then the shape will change
- Continuity Equation
  - Provides relationship between horizontal and vertical motions



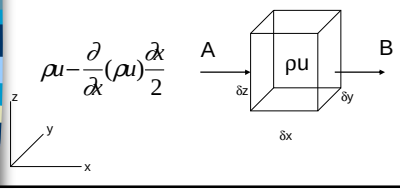
Continuity Equation

- Consider:
  - A cube fixed in space thru which a fluid flows
  - Mass is continually advected through the sides by the fluid motion
    - Net inflow of mass thru sides = Rate of accumulation inside the volume
  - Mass flux = $\frac{\partial M}{\partial t}$

Continuity Equation

- Mass Flux at the center of the cube is ρu
- Using Taylor's expansion about the center, we get in/out flow per unit area through sides A and B

$$\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\partial x}{2}$$



$$\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\partial x}{2}$$

Continuity Equation

- Can get the net rate of flow into the volume due to the x component of velocity
- Then, the net rate of mass inflow per unit volume for all directions is:

$$\frac{\partial M}{\partial t} = -[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)]\delta x\,\delta y\,\delta z$$

Continuity Equation

- ***Mass Divergence*** form of mass continuity

$$\frac{\partial \rho}{\partial t} = -[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)] = -\nabla \cdot (\rho \vec{v})$$

- ***Velocity Divergence*** form of mass continuity

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \cdot \vec{v} = 0$$

The Boussinesq Approximation

- Joseph Boussinesq (1842-1929) noted that we can *safely assume density is constant* <u>except</u> *when it is multiplied by gravity in calculations of pressure*
- The assumption will greatly simplify the equations of motion!

Incompressibility

- If **density is constant**, $\frac{d\rho}{dt} = 0$ then the fluid is **incompressible**
- and $\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \cdot \vec{v} = 0$ becomes:

$$\nabla \cdot \vec{v} = 0$$

  – Density MUST be constant (incompressible)
  – Don't consider vertical length scales so large that hydrostatic pressure causes density variations

Incompressibility

- We can easily use $\nabla \cdot \vec{v} = 0$ to learn about vertical motions
- Expanded:

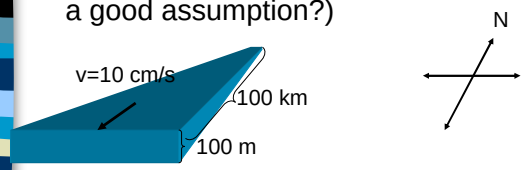
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

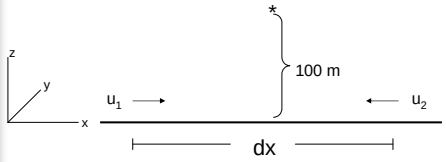
Example - Ocean

- We have an offshore wind, and must solve for the resultant upwelling (i.e. vertical velocity)
- Assume incompressible flow (why is this a good assumption?)



Example - Atmosphere

- Low-level convergence (1m/s each), solve for the vertical velocity 100 m from the surface if v = 0 and dx = 10 m.



Notes

- Conservation laws applied to flow in the atmosphere and ocean lead to equations of motion
- Flow in the ocean can be assumed to be incompressible except when describing sound
- Density can be assumed to be constant except when multiplied by gravity
  - Boussinesq approximation
- Conservation of mass leads to the continuity equation, which has an especially simple form for an incompressible fluid (why?)