**Abbreviated derivation of the quasi-geostrophic omega and height-tendency equations**
SO441 Synoptic Meteorology

1. Start with the basic governing equations of fluid motion (Navier-Stokes):

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - \omega\frac{\partial u}{\partial p} - \frac{\partial \Phi}{\partial x} + fv \qquad (1)$$

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - \omega\frac{\partial v}{\partial p} - \frac{\partial \Phi}{\partial y} - fu \qquad (2)$$

Where the variables (and constants) are defined as follows:

$u$ – east-west velocity component
$v$ – north-south velocity component
$\omega$ – vertical velocity component in pressure coordinates $\omega = \frac{dp}{dt} = \left(\frac{\partial p}{\partial z}\right)\left(\frac{\partial z}{\partial t}\right) = (-\rho g)(w)$
$w$ – vertical velocity component
$g$ – gravity, a constant
$\rho$ – air density
$\Phi$ – geopotential height, $\Phi = gz$
$f$ – Coriolis component, $f = 2\Omega\sin\theta$
$\Omega$ - angular velocity of the Earth, 7.29 x $10^{-5}$ $s^{-1}$, a constant

The horizontal geostrophic wind vector can be written as $\vec{V}_g = u_g\hat{i} + v_g\hat{j}$ and the gradient of the Coriolis component as $\beta = \frac{\partial f}{\partial y}$.

Under a set of restrictions (small Rossby number, adiabatic and frictionless flow, horizontal and uniform static stability, and hydrostatic balance), equations (1) and (2) become the following:

$$\frac{\partial u_g}{\partial t} + u_g\frac{\partial u_g}{\partial x} + v_g\frac{\partial u_g}{\partial y} - f_0 v_{ag} - \beta y v_g = 0 \qquad (3)$$

$$\frac{\partial v_g}{\partial t} + u_g\frac{\partial v_g}{\partial x} + v_g\frac{\partial v_g}{\partial y} + f_0 u_{ag} + \beta y u_g = 0 \qquad (4)$$

where the subscript "g" indicates the geostrophic wind component, "ag" the ageostrophic wind component, $f_0$ the (now constant) Coriolis component, and $\beta$ the change in Coriolis with latitude.

By taking $\frac{\partial}{\partial x}(\text{equation 4}) - \frac{\partial}{\partial y}(\text{equation 3})$, we get

$$\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \cdot \nabla\left(\zeta_g + f\right) + f_0 \frac{\partial \omega}{\partial p}. \qquad (5)$$

Where $\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}$ is the geostrophic relative vorticity component. If $\zeta_g > 0$, relative vorticity is *positive*, which means for the Northern Hemisphere, the spin is cyclonic and clockwise (and for the Southern Hemisphere, the spin is anticyclonic and still clockwise).

Remember that $\zeta_g = \frac{g}{f_0}\nabla^2 Z = \frac{1}{f_0}\nabla^2 \Phi$.

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Now turn to the Thermodynamic Energy Equation:

$$\frac{DT}{Dt} = \frac{\sigma p}{R}\omega$$

where $T$ is temperature, $p$ pressure, $R$ the atmospheric gas constant 287 J kg$^{-1}$ K$^{-1}$, and $\sigma$ the static stability parameter $\sigma = -\frac{RT}{p}\frac{d\ln\theta}{dp}$, where $\theta$ is potential temperature of the air. Expanding out the material derivative gives

$$\left(\frac{\partial}{\partial t} + \vec{V}_g \cdot \nabla\right)T = \frac{\sigma p}{R}\omega$$

The equation of state for air, $p = \rho RT$, can be rewritten as $\frac{1}{\rho} = \frac{RT}{p}$.

Hydrostatic balance, $\frac{\partial p}{\partial z} = -\rho g$, can be rewritten as $\partial p = -\rho g \partial z$ and then combined with the definition of geopotential to give $\partial p = -\rho \partial \Phi$. Rearranging this gives $\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho}$.

Return to the thermodynamic energy equation, divide by $-p/R$ to get $\left(\frac{\partial}{\partial t} + \vec{V}_g \cdot \nabla\right)\frac{-RT}{p} = -\sigma\omega$

Since $\frac{-RT}{p} = \frac{\partial \Phi}{\partial p}$, the thermodynamic energy equation becomes $\left(\frac{\partial}{\partial t} + \vec{V}_g \cdot \nabla\right)\frac{\partial \Phi}{\partial p} = -\sigma\omega$.

Distribute the material derivative operator onto $\frac{\partial \Phi}{\partial p}$ to get $\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial p}\right) + \vec{V}_g \cdot \nabla\left(\frac{\partial \Phi}{\partial p}\right) = -\sigma\omega$.

Rearrange the order of partial derivatives in the first term on the left-hand side to get

$$\frac{\partial}{\partial p}\left(\frac{\partial \Phi}{\partial t}\right) + \vec{V}_g \cdot \nabla\left(\frac{\partial \Phi}{\partial p}\right) = -\sigma\omega.$$

Define a new variable, called the "Height Tendency": $\chi = \frac{\partial \Phi}{\partial t}$. The thermodynamic energy equation now becomes

$$\frac{\partial \chi}{\partial p} + \vec{V}_g \cdot \nabla\left(\frac{\partial \Phi}{\partial p}\right) = -\sigma\omega \qquad (6)$$

Going back to Equation (5), $\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \cdot \nabla\left(\zeta_g + f\right) + f_0 \frac{\partial \omega}{\partial p}$, and substituting for $\zeta_g = \frac{1}{f_0}\nabla^2\Phi$, you get $\frac{\partial}{\partial t}\left(\frac{\nabla^2\Phi}{f_0}\right) = -\vec{V}_g \cdot \nabla\left(\zeta_g + f\right) + f_0 \frac{\partial \omega}{\partial p}$. Rearranging the differentiation, multiplying both sides by $f_0$, and substituting for height-tendency gives:

$$\nabla^2 \chi = -f_0 \vec{V}_g \cdot \nabla\left(\zeta_g + f\right) + f_0^2 \frac{\partial \omega}{\partial p} \qquad (7)$$

Eliminating $\chi$ between equations (6) and (7) gives the **quasi-geostrophic omega equation**.

$$\underbrace{\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)}_{A}\omega = \underbrace{-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \cdot \nabla_p\left(\zeta_g + f\right)\right]}_{B} + \underbrace{\frac{R}{\sigma p}\left[-\nabla_p^2\left(-\vec{V}_g \cdot \nabla_p T\right)\right]}_{C}.$$

The terms in the QG-omega equation are as follows:

$$\underbrace{\left(\nabla_p^2 + \frac{f_0^{\,2}}{\sigma}\frac{\partial^2}{\partial p^2}\right)}_{A}\omega = \underbrace{-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \cdot \nabla_p\left(\zeta_g + f\right)\right]}_{B} + \underbrace{\frac{R}{\sigma p}\left[-\nabla_p^2\left(-\vec{V}_g \cdot \nabla_p T\right)\right]}_{C}$$

**Term A**: A 3-dimensional Laplacian operator. Allows Terms B and C to be called "forcing terms". To solve for omega, would need to "invert" the Laplacian. But note that for wavy fields, a Laplacian operator causes the sign of whatever it operates on to be the opposite of the forcing terms.

**Term B**: Differential geostrophic relative vorticity advection. Note the derivative, $\frac{\partial}{\partial p}$, is a pressure derivative. That means for an air parcel moving from 1000 mb to 500 mb, the denominator of $\frac{\partial}{\partial p}$ is negative.

**Term C**: Laplacian of temperature advection by the geostrophic wind.

<u>Interpretation of the QG-omega equation</u>.
It is very important to note the sign of omega and the sign of the forcing terms. The two general principles are as follows:

1. Differential vorticity advection *increasing with height* (i.e., at 500 mb there is <u>more</u> positive vorticity advection than there is at 700 mb) produces rising motion between the two levels.

2. Warm-air advection at a certain level (850 mb, for example) produces rising motion at that level.

It is imperative that you know how to demonstrate each of these two principles using the terms in the QG-omega equation. Note that both Term B and Term C have negative signs, as well as derivatives.

Eliminating $\omega$ between equations (6) and (7) gives the **quasi-geostrophic height tendency equation**.

$$\underbrace{\left(\nabla_p^2 + \frac{f_0^{\,2}}{\sigma}\frac{\partial^2}{\partial p^2}\right)}_{A}\chi = \underbrace{f_0\left[-\vec{V}_g \cdot \nabla_p\left(\zeta_g + f\right)\right]}_{B} \underbrace{-\frac{f_0^{\,2}}{\sigma}\frac{\partial}{\partial p}\left(\frac{R}{p}\left(-\vec{V}_g \cdot \nabla_p T\right)\right)}_{C}.$$

The terms in the QG height-tendency equation are as follows:

**Term A**: Like in the QG-omega equation, this term is a 3-dimensional Laplacian operator. Allows Terms B and C to be called "forcing terms". To solve for chi, would need to "invert" the Laplacian. But note that for wavy fields, a Laplacian operator causes the sign of whatever it operates on to be the opposite of the forcing terms. So if Term B is positive, chi would be negative.

**Term B**: Geostrophic relative vorticity advection. Note this is just single-level vorticity advection (i.e., the $\frac{\partial}{\partial p}$ derivative, which was present in the QG-omega equation, is gone). If, at 500 mb, there is positive vorticity advection, the sign of chi would be negative, so heights would fall at that location.

**Term C**: Differential temperature advection. Note here the presence of the $\frac{\partial}{\partial p}$ derivative, which notes that this term must be examined in the vertical.

<u>Interpretation of the QG height-tendency equation</u>.
It is very important to note the sign of chi and the sign of the forcing terms. The two general principles are as follows:

1. Cyclonic (positive) geostrophic vorticity advection at 500 mb (positive Term B) produces falling geopotential heights at that level (negative chi).

2. Warm-air advection centered at 700 mb (so negative $\frac{\partial}{\partial p}\left(\frac{R}{p}\left(-\vec{V}_g \cdot \nabla_p T\right)\right)$ and positive Term C from 850 to 700 mb, and positive $\frac{\partial}{\partial p}\left(\frac{R}{p}\left(-\vec{V}_g \cdot \nabla_p T\right)\right)$ and negative Term C from 700 mb to 500 mb) produces falling geopotential heights at 850 mb (negative chi) and height rises at 500 mb (positive chi). Cold-air advection centered at 700 mb would produce the opposite result: height rises at the surface and height falls aloft.

It is imperative that you know how to demonstrate each of these two principles using the terms in the QG height-tendency equation.