Problem 1.1

$$(x^2+y^2)\cos^2\psi + z^2\cot^2\psi = a^2$$

$$(x^2+y^2)\sin^2\cos^2\psi + z^2\cos^2\psi = a^2\sin^2\psi$$

$$(x^2+y^2)\cos^2\psi - (x^2+y^2)\cos^4\psi + z^2\cos^2\psi = a^2 - a^2\cos^2\psi$$

$$(x^2+y^2)\cos^4\psi - (x^2+y^2+z^2+a^2)\cos^2\psi + a^2 = 0$$

$$\cos^2\psi = \frac{x^2+y^2+z^2+a^2 \pm \sqrt{(x^2+y^2+z^2+a^2) - 4a^2(x^2+y^2)}}{2(x^2+y^2)}$$

$$\psi = \cos^{-1}\left\{\frac{x^2+y^2+z^2+a^2 \pm \sqrt{(x^2+y^2+z^2+a^2) - 4a^2(x^2+y^2)}}{2(x^2+y^2)}\right\}^{\frac{1}{2}}$$

$$(x^2+y^2) + \frac{z^2}{\sin^2\psi} - \frac{a^2}{\cos^2\psi} = 0$$

$$2x + \frac{-2z^2\cos\psi}{\sin^3\psi}\frac{\partial\psi}{\partial x} + \frac{-2a^2\sin\psi}{\cos^3\psi}\frac{\partial\psi}{\partial x} = 0$$

$$\frac{\partial\psi}{\partial x} = \frac{x\sin^3\psi\cos^3\psi}{z^2\cos^4\psi + a^2\sin^4\psi}$$

$$z^2\cos^2\psi = a^2\sin^2\psi - (x^2+y^2)\sin^2\psi\cos^2\psi$$

$$z^2\cos^4\psi = a^2\sin^2\psi\cos^2\psi - (x^2+y^2)\sin^2\psi\cos^4\psi$$

$$z^2\cos^4\psi + a^2\sin^4\psi = a^2\sin^2\psi(\cos^2\psi+\sin^2\psi) - (x^2+y^2)\sin^2\psi\cos^2\psi$$

$$z^2\cos^4\psi + a^2\sin^4\psi = a^2\sin^2\psi - (x^2+y^2)\sin^2\psi\cos^4\psi$$

$$\frac{\partial\psi}{\partial x} = \frac{x\sin^3\psi\cos^3\psi}{a^2\sin^2\psi - (x^2+y^2)\sin^2\psi\cos^4\psi} = \frac{x\sin\psi\cos^3\psi}{a^2 - (x^2+y^2)\cos^4\psi}$$

$$\frac{\partial\psi}{\partial y} = \frac{y\sin\psi\cos^3\psi}{a^2 - (x^2+y^2)\cos^4\psi}$$

$$\frac{2z}{\sin^2\psi} - \frac{2z^2\cos^{\psi}}{\sin^3\psi}\frac{\partial\psi}{\partial z} - \frac{2a^2\sin\psi}{\cos^3\psi}\frac{\partial\psi}{\partial z} = 0$$

$$\frac{\partial\psi}{\partial z} = \frac{z\sin\psi\cos^3\psi}{z^2\cos^4\psi + a^2\sin^4\psi} = \frac{z}{x\sin^2\psi}\frac{\partial\psi}{\partial x}$$

$$\frac{\partial \psi}{\partial z} = \frac{z \cos^3 \psi}{\sin \psi (a^2 - (x^2 + y^2) \cos^4 \psi)}$$

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 = \left(x^2 + y^2 + \frac{z^2}{\sin^4 \psi}\right) \left(\frac{\sin \psi \cos^3 \psi}{a^2 - (x^2 + y^2) \cos^4 \psi}\right)^2$$

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 = \left(x^2 + y^2 + \frac{z^2}{\sin^2 \psi} + \frac{z^2 - z^2 \sin^2 \psi}{\sin^4 \psi}\right) \left(\frac{\sin \psi \cos^3 \psi}{a^2 - (x^2 + y^2) \cos^4 \psi}\right)^2$$

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 = \left(\frac{a^2}{\cos^2 \psi} + \frac{z^2 \cos^2 \psi}{\sin^4 \psi}\right) \left(\frac{\sin \psi \cos^3 \psi}{a^2 - (x^2 + y^2) \cos^4 \psi}\right)^2$$

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 = \frac{a^2 \sin^4 \psi + z^2 \cos^4 \psi}{\sin^4 \psi \cos^2 \psi} \left(\frac{\sin \psi \cos^3 \psi}{a^2 - (x^2 + y^2) \cos^4 \psi}\right)^2$$

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 = \frac{a^2 - (x^2 + y^2) \cos^4 \psi}{\sin^2 \psi \cos^2 \psi} \left(\frac{\sin \psi \cos^3 \psi}{a^2 - (x^2 + y^2) \cos^4 \psi}\right)^2$$

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 = \frac{\cos^4 \psi}{a^2 - (x^2 + y^2) \cos^4 \psi}$$

$$\cos \alpha = \frac{\frac{\partial \psi}{\partial x}}{\sqrt{\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2}} = \frac{\frac{x \sin \psi \cos^3 \psi}{a^2 - (x^2 + y^2) \cos^4 \psi}}{\sqrt{\frac{\cos^4 \psi}{a^2 - (x^2 + y^2) \cos^4 \psi}}}$$

$$\cos \alpha = x \sin \psi \cos \psi \{a^2 - (x^2 + y^2) \cos^4 \psi\}^{-\frac{1}{2}}$$

$$\sin^2 \psi = 1 - \cos^2 \psi = \frac{x^2 + y^2 - z^2 - a^2 \mp \sqrt{(x^2 + y^2 + z^2 + a^2)^2 - 4a^2(x^2 + y^2)}}{2(x^2 + y^2)}$$

$$\cos \alpha = \frac{x \sqrt{(x^2 + y^2)^2 - \left\{z^2 + a^2 \pm \sqrt{(x^2 + y^2 + z^2 + a^2) - 4a^2(x^2 + y^2)}\right\}^2}}{\sqrt{4a^2(x^2 + y^2)^2 - (x^2 + y^2)\left\{x^2 + y^2 + z^2 + a^2 \pm \sqrt{(x^2 + y^2 + z^2 + a^2)^2 - 4a^2(x^2 + y^2)}\right\}^2}}$$

$$\cos \beta = \frac{y}{x} \cos \alpha$$

$$\cos \gamma = \frac{z}{x \sin^2 \psi} \cos \alpha$$

$$\frac{\partial \psi}{\partial x} = \frac{x \sin \psi \cos^3 \psi}{a^2 - (x^2 + y^2) \cos^4 \psi}$$

$$\frac{\partial^2\psi}{\partial x^2} = \left\{a^2 - (x^2+y^2)\cos^4\psi\right\}^{-2}\left(\left\{\sin\psi\cos^3\psi + x\cos^4\psi\left(\frac{\partial\psi}{\partial x}\right) - 3x\sin^2\psi\cos^2\psi\left(\frac{\partial\psi}{\partial x}\right)\right\}\left\{a^2 - (x^2+y^2)\cos^4\psi\right\}\right.$$

$$\left.-x\sin\psi\cos^3\psi\left\{-2x\cos^4\psi + 4(x^2+y^2)\sin\psi\cos^3\psi\left(\frac{\partial\psi}{\partial x}\right)\right\}\right)$$

$$\frac{\partial^2\psi}{\partial x^2} = \left\{a^2 - (x^2+y^2)\cos^4\psi\right\}^{-2}\left\{a^2\sin\psi\cos^3\psi + (x^2-y^2)\sin\psi\cos^7\psi\right.$$

$$\left.+x\left(\frac{\partial\psi}{\partial x}\right)\left\{a^2\cos^4\psi - 3a^2\sin^2\psi\cos^2\psi - (x^2+y^2)\cos^6\psi\right\}\right\}$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \left\{a^2 - (x^2+y^2)\cos^4\psi\right\}^{-2}\left\{2a^2\sin\psi\cos^3\psi + \left(x\left(\frac{\partial\psi}{\partial x}\right) + y\left(\frac{\partial\psi}{\partial y}\right)\right)\right.$$

$$\left.\times\left\{a^2\cos^4\psi - 3a^2\sin^2\psi\cos^2\psi - (x^2+y^2)\cos^6\psi\right\}\right\}$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \left\{a^2 - (x^2+y^2)\cos^4\psi\right\}^{-2}\left(2a^2\sin\psi\cos^3\psi +\right.$$

$$\left.\frac{(x^2+y^2)\sin\psi\cos^3\psi}{a^2-(x^2+y^2)\cos^4\psi}\ \left\{a^2\cos^4\psi - 3a^2\sin^2\psi\cos^2\psi - (x^2+y^2)\cos^6\psi\right\}\right)$$

$$\frac{\partial^2\psi}{\partial z^2} = \left\{a^2 - (x^2+y^2)\cos^4\psi\right\}^{-2}\left\{(a^2 - (x^2+y^2)\cos^4\psi)\left(\frac{\cos^3\psi}{\sin\psi} - \frac{3z\sin\psi\cos^2\psi}{\sin\psi}\frac{\partial\psi}{\partial z} - \frac{z\cos^4\psi}{\sin^2\psi}\frac{\partial\psi}{\partial z}\right)\right.$$

$$\left.-\frac{z\cos^3\psi}{\sin\psi}\left(4(x^2+y^2)\sin\psi\cos^3\psi\,\frac{\partial\psi}{\partial z}\right)\right\}$$

$$\frac{\partial^2\psi}{\partial z^2} = \left\{a^2 - (x^2+y^2)\cos^4\psi\right\}^{-2}\left\{(a^2-(x^2+y^2)\cos^4\psi)\left(\frac{\cos^3\psi}{\sin\psi}\right) - z\frac{\partial\psi}{\partial z}\left((a^2 - (x^2+y^2)\cos^4\psi)\left(-3\cos^2\psi - \frac{\cos^4\psi}{\sin^2\psi}\right)\right.\right.$$

$$\left.\left.-4(x^2+y^2)\cos^6\psi\right)\right\}$$

$$z\frac{\partial\psi}{\partial z} = \frac{z^2}{x\sin^2\psi}\frac{\partial\psi}{\partial x} = \left\{-(x^2+y^2) + \frac{a^2}{\cos^2\psi}\right\}\frac{1}{x}\frac{\partial\psi}{\partial x}$$

$$z\frac{\partial\psi}{\partial z} = \left\{-(x^2+y^2) + \frac{a^2}{\cos^2\psi}\right\}\frac{\sin\psi\cos^3\psi}{a^2-(x^2+y^2)\cos^4\psi}$$

$$z\frac{\partial\psi}{\partial z} = \left\{-(x^2+y^2)\cos^4\psi + a^2\cos^2\psi\right\}\frac{\sin\psi}{\cos\psi(a^2-(x^2+y^2)\cos^4\psi)}$$

$$z\frac{\partial\psi}{\partial z} = \left\{-(x^2+y^2)\cos^4\psi - a^2 + a^2 + a^2\cos^2\psi\right\}\frac{\sin\psi}{\cos\psi(a^2-(x^2+y^2)\cos^4\psi)}$$

$$z\frac{\partial\psi}{\partial z} = \frac{\sin\psi}{\cos\psi} - \frac{a^2\sin^3\psi}{\cos\psi(a^2-(x^2+y^2)\cos^4\psi)}$$

$$\frac{\partial^2\psi}{\partial z^2} = \left\{a^2 - (x^2+y^2)\cos^4\psi\right\}^{-2} \left\{ (a^2 - (x^2+y^2)\cos^4\psi)\left(\frac{\cos^3\psi}{\sin\psi}\right) - \left(\frac{\sin\psi}{\cos\psi} - \frac{a^2\sin^3\psi}{\cos\psi(a^2-(x^2+y^2)\cos^4\psi)}\right)\right.$$

$$\left. \times\left((a^2-(x^2+y^2)\cos^4\psi)\left(-3\cos^2\psi - \frac{\cos^4\psi}{\sin^2\psi}\right) - 4(x^2+y^2)\cos^6\psi\right)\right\}$$

$$\frac{\partial^2\psi}{\partial z^2} = \left\{a^2 - (x^2+y^2)\cos^4\psi\right\}^{-2} \left\{ (a^2 - (x^2+y^2)\cos^4\psi)\left(\frac{\cos^3\psi}{\sin\psi} - 3\sin\psi\cos\psi - \frac{\cos\psi}{\sin\psi}\right)\right.$$

$$\left. -4(x^2+y^2)\sin\psi\cos\psi + a^2\sin^3\psi\left(3\cos\psi + \frac{\cos^3\psi}{\sin^2\psi}\right) + 4a^2\frac{(x^2+y^2)\sin^3\cos^5\psi}{(a^2-(x^2+y^2)\cos^4\psi)}\right\}$$

$$\frac{\partial^2\psi}{\partial z^2} = \left\{a^2 - (x^2+y^2)\cos^4\psi\right\}^{-2} \left\{ (a^2 - (x^2+y^2)\cos^4\psi)(-3\sin\psi\cos\psi) - 4(x^2+y^2)\sin\psi\cos^5\psi + 3a^2\sin^3\psi\cos\psi\right.$$

$$\left. -a^2\sin\psi\cos^3\psi + 4a^2\frac{(x^2+y^2)\sin^3\cos^5\psi}{(a^2-(x^2+y^2)\cos^4\psi)}\right\}$$

$$\frac{\partial^2\psi}{\partial z^2} = \left\{a^2 - (x^2+y^2)\cos^4\psi\right\}^{-2} \left\{ a^2(3\sin^3\psi\cos\psi - 3\sin\psi\cos\psi + \sin\psi\cos^3\psi)\right.$$

$$\left. -(x^2+y^2)\cos^4\psi(-4\sin\psi\cos\psi - 3\sin\psi\cos\psi) + 4a^2\frac{(x^2+y^2)\sin^3\cos^5\psi}{(a^2-(x^2+y^2)\cos^4\psi)}\right\}$$

$$\frac{\partial^2\psi}{\partial z^2} = \frac{\sin\psi\cos^3\psi}{\{a^2-(x^2+y^2)\cos^4\psi\}^{-2}}\left\{ -2a^2 - (x^2+y^2)\cos^4\psi\cos^2\psi + 4a^2\frac{(x^2+y^2)\sin^3\cos^5\psi}{(a^2-(x^2+y^2)\cos^4\psi)}\right\}$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{\sin\psi\cos^3\psi}{\{a^2-(x^2+y^2)\cos^4\psi\}^{-2}}\left\{ 2a^2 + (x^2+y^2)\cos^4\psi - 4a^2\frac{(x^2+y^2)\sin^3\cos^5\psi}{(a^2-(x^2+y^2)\cos^4\psi)}\right.$$

$$\left. -2a^2 - (x^2+y^2)\cos^4\psi\cos^2\psi + 4a^2\frac{(x^2+y^2)\sin^3\cos^5\psi}{(a^2-(x^2+y^2)\cos^4\psi)}\right\} = 0$$

The $\psi =$constant surfaces are of the type

$$(x^2+y^2)\alpha^2 + z^2\beta^2 = a^2$$

$$\frac{x^2}{\beta^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\alpha^2} = c^2$$

The $z = 0$ cross-section is a circle; the $x = 0$ and $y = 0$ cross-sections are ellipses. The surface is an ellipsoid of revolution around the z axis. The ellopsoids approach a sphere as $\psi \to \pi/2$. The $\psi = 0$ "surface" is the disk $x^2 + y^2 \leq a^2$. The $\psi = \pi$ surface is identical to the $\psi = 0$ surface.

Problem 1.2

$$\left\{\sqrt{x^2+y^2}-\psi\right\}^2+z^2=\psi^2-a^2 \qquad a<\psi<\infty$$

$$x^2+y^2-2\psi\sqrt{x^2+y^2}+\psi^2+z^2=\psi^2-a^2$$

$$x^2+y^2+z^2+a^2=2\psi\sqrt{x^2+y^2}$$

$$\psi=\frac{1}{2}\frac{x^2+y^2+z^2+a^2}{\sqrt{x^2+y^2}}=\frac{1}{2}\frac{\rho^2+z^2+a^2}{\rho}$$

In the $y=0$ plane

$$\psi=\frac{1}{2}\frac{x^2+z^2+a^2}{x}$$

For $\psi=$constant

$$0=x^2-2x\psi+z^2+a^2$$

$$0=(x-\psi)^2+z^2+a^2-\psi^2$$

$$0=\bar{x}^2+z^2+a^2-\psi^2$$

$$\bar{x}^2+z^2=\psi^2-a^2$$

This curve is a circle of radius $\sqrt{\psi^2-a^2}$ centered at $x=\psi$. Since ψ is symmetric in x and y, the surface $\psi=$constant is the surface of revolution of this circle around the z axis.

In the limiting case $\psi=0$, $a=0$ also and $x=z=0$. The "surface" is the point at the origin. For $\psi\to\infty$, $z\to\infty$ for finite $x^2+y^2>0$ and $x^2+y^2\to\infty$ for finite z. Also, $\psi\to\infty$ for finite z at $x=y=0$. The "surface' $\psi\to\infty$ is the union of the sphere $R=\sqrt{x^2+y^2+z^2}\to\infty$ and the z axis.

$$\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial\rho}\frac{\partial\rho}{\partial x}=\left(\frac{1}{2}-\frac{z^2+a^2}{2\rho^2}\right)\frac{x}{\rho}$$

$$\frac{\partial\psi}{\partial y}=\frac{\partial\psi}{\partial\rho}\frac{\partial\rho}{\partial y}=\left(\frac{1}{2}-\frac{z^2+a^2}{2\rho^2}\right)\frac{y}{\rho}$$

$$\frac{\partial\psi}{\partial z}=\frac{z}{\rho}$$

$$\cos\alpha=\frac{\frac{\partial\psi}{\partial x}}{\sqrt{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2+\left(\frac{\partial\psi}{\partial z}\right)^2}}=\frac{\left(\frac{1}{2}-\frac{z^2+a^2}{2\rho^2}\right)\frac{x}{\rho}}{\sqrt{\left(\frac{1}{2}-\frac{z^2+a^2}{2\rho^2}\right)\frac{x^2+y^2}{\rho^2}+\frac{z^2}{\rho^2}}}$$

$$\cos\alpha=\frac{\left(\frac{1}{2}-\frac{z^2+a^2}{2\rho^2}\right)\frac{x}{\rho}}{\sqrt{\left(\frac{1}{2}-\frac{z^2+a^2}{2\rho^2}\right)+\frac{z^2}{\rho^2}}}$$

$$\cos\beta = \frac{\left(\frac{1}{2} - \frac{z^2+a^2}{2\rho^2}\right)\ \frac{y}{\rho}}{\sqrt{\left(\frac{1}{2} - \frac{z^2+a^2}{2\rho^2}\right) + \frac{z^2}{\rho^2}}}$$

$$\cos\gamma = \frac{\left(\frac{1}{2} - \frac{z^2+a^2}{2\rho^2}\right)\ \frac{z}{\rho}}{\sqrt{\left(\frac{1}{2} - \frac{z^2+a^2}{2\rho^2}\right) + \frac{z^2}{\rho^2}}}$$

$$\frac{\partial^2\psi}{\partial\rho^2} = \frac{z^2+a^2}{\rho^3} \qquad \frac{\partial^2\psi}{\partial z^2} = \frac{1}{\rho}$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\,\frac{\partial\psi}{\partial\rho} + \frac{\partial^2\psi}{\partial z^2} = \frac{z^2+a^2}{\rho^3} + \frac{1}{\rho}\left(\frac{1}{z} - \frac{z^2+a^2}{2\rho^2}\right) + \frac{1}{\rho^2}$$

$$\nabla^2\psi = \frac{z^2+a^2}{2\rho^3} + \frac{3}{2\rho} \neq 0$$

Problem 1.3

$$\frac{dx}{F_x} = \frac{dy}{F_y} = \frac{dz}{F_z}$$

$$F_x = 2zx \qquad F_y = 2zy \qquad F_z = a^2 + z^2 - x^2 - y^2$$

$$F_x\,dy = F_y\,dx$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\ln\left(\frac{y}{y_0}\right) = \ln\left(\frac{x}{x_0}\right)$$

$$\frac{y}{x} = \frac{y_0}{x_0} = \text{constant}$$

$$\frac{y}{x} = \tan\phi$$

$$F_x\,dz = F_z\,dx$$

$$2zx\,dz = (a^2 + z^2 - x^2 - y^2)\,dx$$

$$2zx\,dz = (a^2 + z^2 - x^2 - x^2\tan^2\phi)\,dx$$

$$2zx\,dz = \left(a^2 + z^2 - \frac{x^2}{\cos^2\phi}\right)\,dx$$

$$2zx\,dz - z^2\,dx = \left(a^2 - \frac{x^2}{\cos^2\phi}\right)\,dx$$

$$x\,d(z^2) - z^2\,dx = \left(a^2 - \frac{x^2}{\cos^2\phi}\right)\,dx$$

$$\frac{x\,d(z^2) - z^2\,dx}{x^2} = \left(\frac{a^2}{x^2} - \frac{1}{\cos^2\phi}\right)\,dx$$

$$d\left(\frac{z^2}{x}\right) = \left(\frac{a^2}{x^2} - \frac{1}{\cos^2\phi}\right)\,dx$$

$$\frac{z^2}{x} - \text{constant} = -\frac{a^2}{x} - \frac{x^2}{\cos^2\phi} = -\frac{a^2}{x} - \frac{x}{\frac{x^2}{x^2+y^2}}$$

$$\frac{z^2 + a^2 + x^2 + y^2}{x} = \text{constant}$$

From the symmetry of x and y,

$$\frac{x}{x^2+y^2+z^2+a^2} = c_1 \qquad \frac{y}{x^2+y^2+z^2+a^2} = c_2 \qquad \tan\phi = \frac{c_2}{c_1}$$

$$\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2+a^2} = \sqrt{c_2^2+c_1^2}$$

$$\frac{x^2+y^2+z^2+a^2}{2a\sqrt{x^2+y^2}} = \frac{1}{2a\sqrt{c_1^2+c_2^2}} = \coth\mu$$

$$\frac{\partial F_z}{\partial y} = -2y \qquad \frac{\partial F_z}{\partial x} = -2x$$

$$\frac{\partial F_y}{\partial x} = 2z \qquad \frac{\partial F_y}{\partial z} = 2y$$

$$\frac{\partial F_x}{\partial y} = 2z \qquad \frac{\partial F_x}{\partial z} = 2x$$

$$F_x\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)+F_y\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)+F_z\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 2zx(-2y-2y)+2zy(2x-(-2x))+(a^2+z^2-x^2-y^2)(2z-2z)$$

$$F_x\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)+F_y\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)+F_z\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = -8xyz+8xyz+0 = 0$$

$$F_x\,dx + F_y\,dy + F_z\,dz = 2zx\,dx + 2zy\,dy + (a^2+z^2-x^2-y^2)\,dz = 0$$

$$0 = z\,d(x^2+y^2) - (x^2+y^2)\,dz + (a^2+z^2)\,dz$$

$$0 = \frac{z\,d(x^2+y^2) - (x^2+y^2)\,dz}{z^2} + \frac{(a^2+z^2)}{z^2}\,dz$$

$$0 = d\left(\frac{x^2+y^2}{z}\right) + \frac{a^2+z^2}{z^2}\,dz$$

$$\text{constant} = \frac{x^2+y^2}{z} - \frac{a^2}{z} + z = \frac{x^2+y^2+z^2-a^2}{z}$$

$$\frac{x^2+y^2+z^2-a^2}{2az} = \frac{\text{constant}}{2a} = \cot\psi$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\nabla\phi = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}, \frac{1}{x}, 0\right) = \frac{(-y,x,0)}{x^2+y^2}$$

$$\mu = \tanh^{-1} \frac{2a\sqrt{x^2+y^2}}{x^2+y^2+z^2+a^2}$$

$$\nabla\mu = \frac{1}{1-\frac{4a^2(x^2+y^2)}{(x^2+y^2+z^2+a^2)^2}} \left(\frac{2ax}{(x^2+y^2+z^2+a^2)\sqrt{x^2+y^2}} - \frac{4ax\sqrt{x^2+y^2}}{(x^2+y^2+z^2+a^2)^2}, \right.$$

$$\left. \frac{2ay}{(x^2+y^2+z^2+a^2)\sqrt{x^2+y^2}} - \frac{4ay\sqrt{x^2+y^2}}{(x^2+y^2+z^2+a^2)^2}, \frac{-4az\sqrt{x^2+y^2}}{(x^2+y^2+z^2+a^2)^2} \right)$$

$$\nabla\mu = \frac{1}{(x^2+y^2+z^2+a^2)^2 - 4a^2(x^2+y^2)} \left(\frac{2ax(z^2+a^2-x^2-y^2)^2}{\sqrt{x^2+y^2}}, \frac{2ay(z^2+a^2-x^2-y^2)^2}{\sqrt{x^2+y^2}}, \frac{-4az(x^2+y^2)}{\sqrt{x^2+y^2}} \right)$$

$$\nabla\phi \cdot \nabla\mu = \left(\frac{1}{x^2+y^2}\right) \left(\frac{1}{(x^2+y^2+z^2+a^2)^2 - 4a^2(x^2+y^2)} \right) \left(\frac{-2ayx(z^2+a^2-x^2-y^2)}{\sqrt{x^2+y^2}} + \frac{2ayx(z^2+a^2-x^2-y^2)}{\sqrt{x^2+y^2}} + 0 \right)$$

$$\nabla\phi \cdot \nabla\mu = 0$$

$$\psi = \tan^{-1} \frac{2az}{x^2+y^2+z^2-a^2}$$

$$\nabla\psi = \frac{1}{1+\frac{4a^2z^2}{(x^2+y^2+z^2-a^2)^2}} \left(-\frac{4axz}{(x^2+y^2+z^2-a^2)^2}, -\frac{4ayz}{(x^2+y^2+z^2-a^2)^2}, \frac{2a(x^2+y^2+z^2-a^2)-4az^2}{(x^2+y^2+z^2-a^2)^2} \right)$$

$$\nabla\psi = \frac{1}{(x^2+y^2+z^2-a^2)^2 - 4a^2z^2} \left(-4axz, -4ayz, 2a(x^2+y^2-z^2-a^2) \right)$$

$$\nabla\phi \cdot \nabla\psi = \left(\frac{1}{x^2+y^2}\right) \left(\frac{1}{(x^2+y^2+z^2-a^2)^2 - 4a^2z^2} \right) (4axyz - 4axyz + 0)$$

$$\nabla\phi \cdot \nabla\psi = 0$$

$$\nabla\psi \cdot \nabla\mu = \left(\frac{1}{(x^2+y^2+z^2-a^2) - 4a^2z^2} \right) \left(\frac{1}{(x^2+y^2+z^2+a^2)^2 - 4a^2(x^2+y^2)} \right) \times$$

$$\left(\frac{-8a^2x^2z(z^2+a^2-x^2-y^2)}{\sqrt{x^2+y^2}} - \frac{8a^2y^2z(z^2+a^2-x^2-y^2)}{\sqrt{x^2+y^2}} + \frac{-8a^2z(x^2+y^2)(-z^2-a^2+x^2+y^2)}{\sqrt{x^2+y^2}} \right)$$

$$\nabla\psi \cdot \nabla\mu = 0$$

Problem 1.4

$$\frac{dx}{F_x} = \frac{dy}{F_y} = \frac{dz}{F_z}$$

$$F_x = 3xz \qquad F_y = 3yz \qquad F_z = 2z^2 - x^2 - y^2$$

$$\frac{dx}{3xz} = \frac{dy}{3yz}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\ln\left(\frac{x}{x_0}\right) = \ln\left(\frac{y}{y_0}\right)$$

$$\frac{y}{x} = \text{constant} = \tan\phi \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{dx}{3xz} = \frac{dz}{2z^2 - x^2 - y^2} = \frac{dz}{2z^2 - x^2/\cos^2\phi}$$

$$\left(2z^2 - \frac{x^2}{\cos^2\phi}\right) dx = 3xz\,dz$$

$$2z^2\,dx - 3xz\,dz = \frac{x^2}{\cos^2\phi}\,dx$$

$$z^2(2\,dx) - \frac{3}{2}x\,d(z^2) = \frac{x^2}{\cos^2\phi}\,dx$$

$$-\frac{3}{2}x^{7/3}\left\{z^2\left(-\frac{4}{3}x^{-7/3}\,dx\right) + x^{-4/3}\,d(z^2)\right\} = \frac{x^2}{\cos^2\phi}\,dx$$

$$-\frac{3}{2}x^{7/3}\left\{d(z^2x^{-4/3})\right\} = \frac{x^2}{\cos^2\phi}\,dx$$

$$d(z^2x^{-4/3}) = -\frac{2}{3}\frac{x^{-1/3}}{\cos^2\phi}\,dx$$

$$z^2x^{-4/3} + c_1 = -\frac{x^{2/3}}{\cos^2\phi}$$

$$c_1 = z^2x^{-4/3} + \frac{x^{2/3}}{\cos^2\phi} = z^2x^{-4/3} + \frac{x^2+y^2}{x^{4/3}} = \frac{x^2+y^2+z^2}{x^{4/3}}$$

$$c_2 = \frac{x^2+y^2+z^2}{y^{4/3}}$$

$$\frac{1}{c_1} + \frac{1}{c_2} = \frac{x^{4/3} + y^{4/3}}{x^2 + y^2 + z^2}$$

$$\theta = \frac{x^2 + y^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$$3xz\,dx + 3yz\,dy + (2z^2 - y^2 - x^2)\,dz = 0$$

$$\frac{3z}{2}\,d(x^2) + \frac{3z}{2}\,d(y^2) - (y^2 + x^2)\,dz = -2z^2\,dz$$

$$\frac{3}{2}z\,d(x^2 + y^2) - (x^2 + y^2)\,dz = -2z^2\,dz$$

$$\frac{3}{2}z^{5/3}\left\{z^{-2/3}\,d(x^2 + y^2) - \frac{2}{3}z^{-5/3}(x^2 + y^2)\,dz\right\} = -2z^2\,dz$$

$$\frac{3}{2}z^{5/3}\,d\left[(x^2 + y^2)z^{-2/3}\right] = -2z^2\,dz$$

$$d\left[(x^2 + y^2)z^{-2/3}\right] = -\frac{4}{3}x^{1/3}\,dz$$

$$(x^2 + y^2)z^{-2/3} + \text{constant} = -z^{4/3}$$

$$\text{constant} = z^{4/3} + \frac{x^2 + y^2}{z^{2/3}} = \frac{x^2 + y^2 + z^2}{z^{2/3}}$$

$$\psi = \frac{z}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial\psi}{\partial x} = -\frac{3zx}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{3z(x^2 + y^2 + z^2) + 15zx^2}{(x^2 + y^2 + z^2)^{7/2}} = \frac{12zx^2 - 3zy^2 - 3z^3}{(x^2 + y^2 + z^2)^{7/2}}$$

$$\frac{\partial^2\psi}{\partial y^2} = -\frac{3z(x^2 + y^2 + z^2) + 15zy^2}{(x^2 + y^2 + z^2)^{7/2}} = \frac{-3zx^2 + 12zy^2 - 3z^3}{(x^2 + y^2 + z^2)^{7/2}}$$

$$\frac{\partial^2\psi}{\partial z^2} = \frac{\partial}{\partial z}\left\{\frac{1}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3z^2}{(x^2 + y^2 + z^2)^{5/2}}\right\}$$

$$\frac{\partial^2\psi}{\partial z^2} = \frac{\partial}{\partial z}\left\{\frac{x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{5/2}}\right\}$$

$$\frac{\partial^2\psi}{\partial z^2} = -\frac{4z(x^2 + y^2 + z^2) - 5z(x^2 + y^2 - 2z^2)}{(x^2 + y^2 + z^2)^{7/2}} = \frac{-9zx^2 - 9zy^2 + 6z^3}{(x^2 + y^2 + z^2)^{7/2}}$$

$$\nabla^2\psi = (x^2 + y^2 + z^2)^{-7/2}\left\{12zx^2 - 3zy^2 - 3y^3 - 3zx^2 + 12zy^2 - 3z^3 - 9zx^2 - 9zy^2 + 6z^3\right\}$$

$$\nabla^2\psi = (x^2 + y^2 + z^2)^{-7/2}\{0\} = 0.$$

Yes, ψ obeys Laplace's Equation.

Problem 1.5

$$F_x = 2zx \qquad F_y = 2zy \qquad F_z = a^2 + z^2 - x^2 - y^2$$

Complete sphere:

$$\oint \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, r^2 \left\{ \frac{2zx^2 + 2zy^2 + (a^2 + z^2 - x^2 - y^2)z}{\sqrt{x^2 + y^2 + z^2}} \right\}$$

$$\oint \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, r\{2(r\cos\theta)(r^2\sin^2\theta) + (a^2 + r^2\cos^2\theta - r^2\sin^2\theta)r\cos\theta\}$$

$$\oint \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, r\{r^3\sin^2\theta\cos\theta + a^2 r\cos\theta + r^3\cos^3\theta\}$$

$$\oint \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = 2\pi \int_0^{\pi} \{r^4\sin^3\theta\cos\theta\, d\theta + a^2r^2\sin\theta\cos\theta\, d\theta + r^4\cos^3\theta\sin\theta\, d\theta\}$$

$$\oint \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = 2\pi \left\{ r^4\frac{\sin^4\theta}{4} + a^2r^2\frac{\sin^2\theta}{2} - r^4\frac{\cos^4\theta}{4} \right\}\Bigg|_0^{\pi}$$

$$\oint \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = 0$$

$z > 0$ hemisphere:

$$\int \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = 2\pi \left\{ r^4\frac{\sin^4\theta}{4} + a^2r^2\frac{\sin^2\theta}{2} - r^4\frac{\cos^4\theta}{4} \right\}\Bigg|_0^{\frac{\pi}{2}}$$

$$\int \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = \frac{\pi r^4}{2} + \pi a^2 r^2 - \frac{\pi r^4}{2} = \pi a^2 r^2$$

$z < 0$ hemisphere:

$$\int \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = 2\pi \left\{ r^4\frac{\sin^4\theta}{4} + a^2r^2\frac{\sin^2\theta}{2} - r^4\frac{\cos^4\theta}{4} \right\}\Bigg|_{\frac{\pi}{2}}^{\pi}$$

$$\int \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = -\frac{\pi r^4}{2} - \pi a^2 r^2 + \frac{\pi r^4}{2} = -\pi a^2 r^2$$

$z = 0$ plane:

$$\int \mathbf{F} \cdot \hat{\mathbf{n}} = \int_0^{2\pi} d\phi \int_0^{r} r\, dr\, (-a^2 + r^2) = -2\pi \left\{ \frac{a^2r^2}{2} - \frac{r^4}{4} \right\}\Bigg|_0^{r} = -2\pi \left\{ \frac{a^2r^2}{2} - \frac{r^4}{4} \right\}$$

net$=-2\pi a^2 r^2 + \frac{\pi}{2} r^4$.

Problem 1.6

$$F_x = \frac{x-a}{(x-a)^2+y^2} - \frac{x}{x^2+y^2} \qquad F_y = \frac{y}{x^2+y^2} - \frac{y}{(x-a)^2+y^2} \qquad F_z = 0$$

$$\frac{\partial F_x}{\partial y} = \frac{-2y(x-a)}{\left[(x-a)^2+y^2\right]^2} + \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial F_y}{\partial x} = \frac{2y(x-a)}{\left[(x-a)^2+y^2\right]^2} - \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} =$$

$$\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = \frac{\partial F_y}{\partial z} - \frac{\partial F_z}{\partial y} = 0$$

$$F_x = 2zx \qquad F_y = 2zy \qquad F_z = a^2 + z^2 - x^2 - y^2$$

$$\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = 0$$

$$\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} = -2x - 2x = -4x$$

$$\frac{\partial F_y}{\partial z} - \frac{\partial F_z}{\partial y} = 2y - (-2y) = 4y$$

$$\nabla \times \mathbf{F} = \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$\nabla \times \mathbf{F} = -4y\mathbf{i} - 4z\mathbf{j}$$

$\phi = 0$ implies $y = x \tan\phi = 0$. The path lies entirely in the y plane so that $\mathbf{n} = \mathbf{j}$.

$$\int (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA = \int (-4z)\, dA = \int_{\frac{-a}{\sqrt{1-c^2}}}^{\frac{a}{\sqrt{1-c^2}}} dx \int_{-\sqrt{(c^2-1)x^2-a^2}}^{\sqrt{(c^2-1)x^2+a^2}} -4z\, dz$$

$$\int (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA = \int_{\frac{-a}{\sqrt{1-c^2}}}^{\frac{a}{\sqrt{1-c^2}}} dx \cdot (-2z^2)\Bigg|_{-\sqrt{(c^2-1)x^2-a^2}}^{\sqrt{(c^2-1)x^2-a^2}} = 0$$

Problem 1.7

$$\lambda = \{z + (x^2 + y^2 + z^2)^{1/2}\}^{1/2} = \{r\cos\theta + r\}^{1/2}$$

$$\mu = \{-z + (x^2 + y^2 + z^2)^{1/2}\}^{1/2} = \{-r\cos\theta + r\}^{1/2}$$

$$\phi = \tan^{-1}\frac{y}{x}$$

$$\lambda^2 = r + r\cos\theta \qquad \mu^2 = r - r\cos\theta$$

$$\lambda^2 + \mu^2 = 2r \qquad \lambda^2 - \mu^2 = 2r\cos\theta$$

$$r = \frac{1}{2}(\lambda^2 + \mu^2) \qquad \theta = \cos^{-1}\frac{\lambda^2 - \mu^2}{\lambda^2 + \mu^2} \qquad \phi = \phi$$

$$\frac{\partial r}{\partial \lambda} = \lambda \qquad \frac{\partial r}{\partial \mu} = \mu \qquad \frac{\partial r}{\partial \phi} = 0$$

$$\begin{aligned}
\frac{\partial \theta}{\partial \lambda} &= \frac{-1}{\sqrt{1 - \left(\frac{\lambda^2-\mu^2}{\lambda^2+\mu^2}\right)^2}}\left\{\frac{2\lambda}{\lambda^2+\mu^2} - \frac{(\lambda^2-\mu^2)(2\lambda)}{(\lambda^2+\mu^2)^2}\right\} \\
&= \frac{-1}{\sqrt{1 - \left(\frac{\lambda^2-\mu^2}{\lambda^2+\mu^2}\right)^2}}\left\{\frac{2\lambda(\lambda^2+\mu^2) - 2\lambda(\lambda^2-\mu^2)}{(\lambda^2+\mu^2)^2}\right\} \\
&= \frac{-4\lambda\mu^2}{(\lambda^2+\mu^2)\sqrt{(\lambda^2+\mu^2)^2 - (\lambda^2-\mu^2)^2}} = \frac{-4\lambda\mu^2}{(\lambda^2+\mu^2)\sqrt{4\lambda^2\mu^2}} \\
&= \frac{-2\mu}{\lambda^2+\mu^2} \\
\frac{\partial \theta}{\partial \mu} &= \frac{2\lambda}{\lambda^2+\mu^2} \\
\frac{\partial \theta}{\partial \phi} &= 0
\end{aligned}$$

$$\frac{\partial \mu}{\partial \lambda} = 0 \qquad \frac{\partial \phi}{\partial \mu} = 0 \qquad \frac{\partial \phi}{\partial \phi} = 1$$

$$\begin{aligned}
g_{\lambda\lambda} &= g_{rr}\left(\frac{\partial r}{\partial \lambda}\right)^2 + g_{\theta\theta}\left(\frac{\partial \theta}{\partial \lambda}\right)^2 + g_{\phi\phi}\left(\frac{\partial \phi}{\partial \lambda}\right)^2 \\
&= \lambda^2 + r^2\frac{4\mu^2}{(\lambda^2+\mu^2)^2} \\
&= \lambda^2 + \frac{1}{4}(\lambda^2+\mu^2)^2\frac{4\mu^2}{(\lambda^2+\mu^2)^2} = \lambda^2 + \mu^2
\end{aligned}$$

$$\begin{aligned}
g_{\mu\mu} &= g_{rr}\left(\frac{\partial r}{\partial \mu}\right)^2 + g_{\theta\theta}\left(\frac{\partial \theta}{\partial \mu}\right)^2 + g_{\phi\phi}\left(\frac{\partial \phi}{\partial \mu}\right)^2 \\
&= \mu^2 + r^2\frac{4\lambda^2}{(\lambda^2+\mu^2)^2} \\
&= \mu^2 + \frac{1}{4}(\lambda^2+\mu^2)^2\frac{4\lambda^2}{(\lambda^2+\mu^2)^2} = \lambda^2 + \mu^2
\end{aligned}$$

$$
\begin{aligned}
g_{\phi\phi} &= g_{rr}\left(\frac{\partial r}{\partial \phi}\right)^2 + g_{\theta\theta}\left(\frac{\partial \theta}{\partial \phi}\right)^2 + g_{\phi\phi}\left(\frac{\partial \phi}{\partial \phi}\right)^2 \\
&= r^2 \sin^2\theta \\
&= \frac{1}{4}(\lambda^2+\mu^2)^2\left\{1 - \frac{(\lambda^2-\mu^2)^2}{(\lambda^2+\mu^2)^2}\right\} = \frac{1}{4}\left\{(\lambda^2+\mu^2)^2 - (\lambda^2-\mu^2)^2\right\} \\
&= \lambda^2\mu^2
\end{aligned}
$$

$$
\sqrt{x^2+y^2+z^2} = \frac{1}{2}(\lambda^2+\mu^2) \qquad z = \frac{1}{2}(\lambda^2-\mu^2) \qquad \frac{y}{x} = \tan\phi
$$

$$
\begin{aligned}
x^2+y^2+z^2 &= \frac{1}{4}(\lambda^2+\mu^2)^2 \\
x^2 + x^2\tan^2\phi + \frac{1}{4}(\lambda^2-\mu^2)^2 &= \frac{1}{4}(\lambda^2+\mu^2)^2 \\
\frac{x^2}{\cos^2\theta} &= \lambda^2\mu^2 \\
x &= \lambda\mu\cos\phi \\
y &= \lambda\mu\sin\theta \\
z &= \frac{1}{2}(\lambda^2-\mu^2)
\end{aligned}
$$

$$
(\nabla\times\mathbf{F})_{ij} = F_{j,i} - F_{i,j}
$$

$$
\begin{aligned}
\nabla^2\psi &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\frac{\partial\psi}{\partial x^j}\right)g^{ij} \\
&= \psi_{,ij}g^{ij} + \frac{g_{,i}}{2g}\psi_{,j}g^{ij} \\
&= g^{\lambda\lambda}\psi_{,\lambda\lambda} + g^{\mu\mu}\psi_{,\mu\mu} + g^{\phi\phi}\psi,\phi\phi + \frac{g_{,\lambda}}{2g}\psi_{,\lambda} + \frac{g_{,\mu}}{2g}\psi_{,\mu}
\end{aligned}
$$

$$
g = g_{\lambda\lambda}g_{\mu\mu}g_{\phi\phi} = \lambda^2\mu^2(\lambda^2+\mu^2)^2
$$

$$
\frac{g_{,\lambda}}{2g} = \frac{1}{2}\left\{\frac{2}{\lambda} + \frac{4\lambda}{\lambda^2+\mu^2}\right\} = \frac{1}{\lambda} + \frac{2\lambda}{\lambda^2+\mu^2} = \frac{3\lambda^2+\mu^2}{\lambda(\lambda^2+\mu^2)}
$$

$$
\frac{g_{,\mu}}{2g} = \frac{3\mu^2+\lambda^2}{\mu(\lambda^2+\mu^2)}
$$

$$
\nabla^2\psi = \frac{1}{\lambda^2+\mu^2}\frac{\partial^2\psi}{\partial\lambda^2} + \frac{1}{\lambda^2+\mu^2}\frac{\partial^2\psi}{\partial\mu^2} + \frac{1}{\lambda^2+\mu^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{3\lambda^2+\mu^2}{\lambda(\lambda^2+\mu^2)}\frac{\partial\psi}{\partial\lambda} + \frac{3\mu^2+\lambda^2}{\mu(\lambda^2+\mu^2)}\frac{\partial\psi}{\partial\mu}
$$

$$
\begin{aligned}
F_\lambda &= F_x\frac{\partial x}{\partial\lambda} + F_y\frac{\partial y}{\partial\lambda} + F_z\frac{\partial z}{\partial\lambda} \\
&= \frac{x/\sqrt{x^2+y^2+z^2}}{z+\sqrt{x^2+y^2+z^2}}(\mu\cos\phi) + \frac{y/\sqrt{x^2+y^2+z^2}}{z+\sqrt{x^2+y^2+z^2}}(\mu\sin\phi) + \frac{1}{\sqrt{x^2+y^2+z^2}}(\lambda) \\
&= \frac{\lambda\mu^2\cos^2\phi/\frac{1}{2}(\lambda^2+\mu^2)}{\frac{1}{2}(\lambda^2-\mu^2)+\frac{1}{2}(\lambda^2+\mu^2)} + \frac{\lambda\mu^2\sin^2\phi/\frac{1}{2}(\lambda^2+\mu^2)}{\frac{1}{2}(\lambda^2-\mu^2)+\frac{1}{2}(\lambda^2+\mu^2)} + \frac{\lambda}{\frac{1}{2}(\lambda^2+\mu^2)} \\
&= \frac{\lambda\mu^2}{\lambda^2(\lambda^2+\mu^2)} + \frac{2\lambda}{\lambda^2+\mu^2} = \frac{\mu^2+2\lambda^2}{\lambda(\lambda^2+\mu^2)}
\end{aligned}
$$

$$
\begin{aligned}
F_\mu &= F_x \frac{\partial x}{\partial \mu} + F_y \frac{\partial y}{\partial \mu} + F_z \frac{\partial z}{\partial \mu} \\
&= = \frac{\lambda^2 + 2\mu^2}{\lambda(\lambda^2+\mu^2)}
\end{aligned}
$$

$$
\begin{aligned}
F_\phi &= F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} + F_z \frac{\partial z}{\partial \phi} \\
&= \frac{x/\sqrt{x^2+y^2+z^2}}{z+\sqrt{x^2+y^2+z^2}}(-\lambda\mu\sin\phi) + \frac{y/\sqrt{x^2+y^2+z^2}}{z+\sqrt{x^2+y^2+z^2}}(\lambda\mu\cos\phi) + \frac{1}{\sqrt{x^2+y^2+z^2}}(0) \\
&= \frac{x/\sqrt{x^2+y^2+z^2}}{z+\sqrt{x^2+y^2+z^2}}(-y) + \frac{y/\sqrt{x^2+y^2+z^2}}{z+\sqrt{x^2+y^2+z^2}}(x) = 0
\end{aligned}
$$

$$
\begin{aligned}
\nabla \cdot \mathbf{F} &= \frac{g^{ij}}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} F_j\right) \\
&= F_{i,j} g^{ij} + \frac{g_{,i}}{2g} F_j g^{ij} \\
&= F_{\lambda,\lambda} g^{\lambda\lambda} + F_{\mu,\mu} g^{\mu\mu} + F_{\phi,\phi} g^{\phi\phi} + \frac{g_{,\lambda}}{2g} F_\lambda g^{\lambda\lambda} + \frac{g_{,\mu}}{2g} F_\mu g^{\mu\mu}
\end{aligned}
$$

$$
F_{\lambda,\lambda} = -\frac{\mu^4}{\lambda^2(\lambda^2+\mu^2)^2}
$$

$$
F_{\mu,\mu} = -\frac{\lambda^4}{\mu^2(\lambda^2+\mu^2)^2}
$$

$$
\begin{aligned}
\nabla \cdot \mathbf{F} &= -\frac{\mu^4}{\lambda^2(\lambda^2+\mu^2)^2} \frac{1}{\lambda^2+\mu^2} + \frac{\mu^2+2\lambda^2}{\lambda(\lambda^2+\mu^2)} \frac{3\lambda^2+\mu^2}{\lambda(\lambda^2+\mu^2)} \frac{1}{\lambda^2+\mu^2} - \frac{\lambda^4}{\mu^2(\lambda^2+\mu^2)^2} \frac{1}{\lambda^2+\mu^2} + \frac{\lambda^2+2\mu^2}{\mu(\lambda^2+\mu^2)} \frac{3\mu^2+\lambda^2}{\mu(\lambda^2+\mu^2)} \frac{1}{\lambda^2+\mu^2} \\
&= \frac{11}{(\lambda^2+\mu^2)^2}
\end{aligned}
$$

Problem 1.8

$$\phi = \tan^{-1} \frac{y}{x} \qquad \mu = \tanh^{-1} \frac{2a\sqrt{x^2 + y^2}}{x^2 + y^2 + z^2 + a^2} \qquad \psi = \tan^{-1} \frac{2az}{x^2 + y^2 + z^2 - a^2}$$

$$y = x \tan\phi \qquad y^2 + x^2 = \frac{x^2}{\cos^2\phi}$$

$$\begin{aligned} x^2 + y^2 + z^2 + a^2 &= 2a \coth\mu \sqrt{x^2 + y^2} \\ \frac{x^2}{\cos^2\phi} + z^2 + a^2 &= 2a \coth\mu \frac{x}{\cos\phi} \\ \frac{x^2}{\cos^2\phi} + z^2 &= 2a \coth\mu \frac{x}{\cos\phi} - a^2 \end{aligned}$$

Problem 1.9

$$\xi = \ln(x^2+y^2) - 2z$$

$$\nabla\xi = \left(\frac{2x}{x^2+y^2}\,,\frac{2y}{x^2+y^2}\,,-2\right)$$

$$\eta = \frac{1}{2}(x^2+y^2) + z$$

$$\nabla\eta = (x, y, 1)$$

$$\phi = \tan^{-1}\frac{y}{x}$$

$$\nabla\phi = \left(\frac{-y}{x^2+y^2}\,,\frac{x}{x^2+y^2}\,,0\right)$$

$$\nabla\xi\cdot\nabla\eta = \frac{2x^2}{x^2+y^2} + \frac{2y^2}{x^2+y^2} - 2 = 0$$

$$\nabla\xi\cdot\nabla\phi = \frac{-2xy}{x^2+y^2} + \frac{2xy}{x^2+y^2} + 0 = 0$$

$$\nabla\eta\cdot\nabla\phi = \frac{-xy}{x^2+y^2} + \frac{xy}{x^2+y^2} + 0 = 0$$

$$\left|\frac{\partial\xi_i}{\partial x_j}\right| = \begin{vmatrix} \frac{2x}{x^2+y^2} & \frac{2y}{x^2+y^2} & -2 \\ x & y & 1 \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = \frac{2x}{x^2+y^2}\left(\frac{-x}{x^2+y^2}\right) + \frac{2y}{x^2+y^2}\left(\frac{-y}{x^2+y^2}\right) - 2 = -2\frac{x^2+y^2+1}{x^2+y^2}$$

$$\left(\frac{\partial x_i}{\partial\xi_j}\right) = \begin{pmatrix} \frac{x}{2(x^2+y^2+1)} & \frac{x}{x^2+y^2+1} & -y \\ \frac{y}{2(x^2+y^2+1)} & \frac{y}{x^2+y^2+1} & x \\ \frac{-x^2-y^2}{2(x^2+y^2+1)} & \frac{1}{x^2+y^2+1} & 0 \end{pmatrix}$$

Problem 1.11

$$\hat{\mathbf{a}}_\theta = \frac{\vec{\mathbf{a}}_\theta}{|\vec{\mathbf{a}}_\theta|} = \frac{\vec{\mathbf{a}}_\theta}{\sqrt{\vec{\mathbf{a}}_\theta \cdot \vec{\mathbf{a}}_\theta}} = \frac{\vec{\mathbf{a}}_\theta}{\sqrt{g^{\theta\theta} a_\theta^2}}$$
$$= \left(0, \frac{1}{\sqrt{g^{\theta\theta}}}, 0\right)$$

$$\begin{aligned}(\vec{\mathbf{a}}_\theta \cdot \nabla)\vec{A} &= g^{\theta\theta}(\vec{\mathbf{a}}_\theta)_\theta \frac{\partial A_k}{\partial \theta} \\ &= \frac{g^{\theta\theta}}{\sqrt{g^{\theta\theta}}} \frac{\partial A_k}{\partial \theta} = \sqrt{g^{\theta\theta}} A_{k,\theta} \\ &= \frac{1}{r} \frac{\partial A_k}{\partial \theta}\end{aligned}$$

$$\begin{aligned}(\vec{\mathbf{a}}_\phi \cdot \nabla)\vec{B} &= g^{\phi\phi}(\vec{\mathbf{a}}_\phi)_\phi \frac{\partial B_k}{\partial \phi} \\ &= \frac{g^{\phi\phi}}{\sqrt{g^{\phi\phi}}} \frac{\partial B_k}{\partial \phi} = \sqrt{g^{\phi\phi}} B_{k,\phi} \\ &= \frac{1}{r \sin\theta} \frac{\partial B_k}{\partial \phi}\end{aligned}$$

$$x = a\cosh\mu\cos\theta\cos\phi \qquad y = a\cosh\mu\cos\theta\sin\phi \qquad z = a\sinh\mu\sin\theta$$

$$\frac{\partial x}{\partial \mu} = a\sinh\mu\cos\theta\cos\phi \qquad \frac{\partial y}{\partial \mu} = a\sinh\mu\cos\theta\sin\phi \qquad \frac{\partial z}{\partial \mu} = a\cosh\mu\sin\theta$$

$$\frac{\partial x}{\partial \theta} = -a\cosh\mu\sin\theta\cos\phi \qquad \frac{\partial y}{\partial \theta}- = a\cosh\mu\sin\theta\sin\phi \qquad \frac{\partial z}{\partial \theta} = a\sinh\mu\cos\theta$$

$$\frac{\partial x}{\partial \phi} = -a\sinh\mu\cos\theta\sin\phi \qquad \frac{\partial y}{\partial \phi} = a\sinh\mu\cos\theta\cos\phi \qquad \frac{\partial z}{\partial \mu} = 0$$

$$\begin{aligned}g_{\mu\mu} &= \left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2 + \left(\frac{\partial z}{\partial \mu}\right)^2 \\ &= a^2\sinh^2\mu\cos^2\theta\cos^2\phi + a^2\sinh^2\mu\cos^2\theta\sin^2\phi + a^2\cosh^2\mu\sin^2\theta \\ &= a^2\sinh^2\mu\cos^2\theta + a^2\cosh^2\mu\sin^2\theta\end{aligned}$$

$$\begin{aligned}g_{\phi\phi} &= \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 \\ &= a^2\sinh^2\mu\cos^2\theta\sin^2\phi + a^2\sinh^2\mu\cos^2\theta\cos^2\phi \\ &= a^2\sinh^2\mu\cos^2\theta\end{aligned}$$

$$\begin{aligned}g_{\theta\theta} &= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \\ &= a^2\cosh^2\mu\sin^2\theta\cos^2\phi + a^2\cosh^2\mu\sin^2\theta\sin^2\phi + a^2\sinh^2\mu\cos^2\theta \\ &= a^2\cosh^2\mu\sin^2\theta + a^2\sinh^2\mu\cos^2\theta \\ &= a^2\sinh^2\mu\sin^2\theta + a^2\sin^2\theta + a^2\sinh^2\mu\cos^2\theta \\ &= a^2\sinh^2\mu + a^2\sin^2\theta\end{aligned}$$

$$(\vec{\mathbf{a}}_\theta \cdot \nabla)\vec{A} = \frac{1}{a\sqrt{\sinh^2 \mu + \sin^2 \theta}} \frac{\partial A_k}{\partial \theta}$$

$$(\vec{\mathbf{a}}_\phi \cdot \nabla)\vec{B} = \frac{1}{a \sinh \mu \cos \theta} \frac{\partial B_k}{\partial \phi}$$

Problem 1.13

$$\{\nabla \times (u\nabla v)\}^i = g^{-1/2}\epsilon^{ijk}(uv_{,k})_{,j}$$

$$\{\nabla \times (u\nabla v)\}^i = g^{-1/2}\epsilon^{ijk}(u_{,j}v_{,k} + uv_{,kj})$$

$$\{\nabla \times (u\nabla v)\}^i = g^{-1/2}\epsilon^{ijk}u_{,j}v_{,k}$$

$$\nabla \times (u\nabla v) = \frac{1}{\sqrt{g}}\left(\frac{\partial u}{\partial \xi_2}\frac{\partial v}{\partial \xi_3} - \frac{\partial u}{\partial \xi_3}\frac{\partial v}{\partial \xi_2}, \frac{\partial u}{\partial \xi_3}\frac{\partial v}{\partial \xi_1} - \frac{\partial u}{\partial \xi_1}\frac{\partial v}{\partial \xi_3}, \frac{\partial u}{\partial \xi_1}\frac{\partial v}{\partial \xi_2} - \frac{\partial u}{\partial \xi_2}\frac{\partial v}{\partial \xi_1}\right)$$

Problem 1.15

$$g_{\theta\theta} = \frac{a^2}{(\cosh\mu - \cos\theta)^2} \quad g_{\phi\phi} = \frac{a^2\sin^2\theta}{(\cosh\mu - \cos\theta)^2} \quad g_{\mu\mu} = g_{\theta\theta}$$

$$\Gamma^{\theta}_{\theta\theta} = \frac{1}{2}g^{\theta\theta}g_{\theta\theta,\theta} = \frac{1}{2}\left(\frac{\cosh\mu - \cos\theta}{a}\right)^2 \left\{\frac{-2a^2\sin\theta}{(\cosh\mu - \cos\theta)^3}\right\} = \frac{-\sin\theta}{\cosh\mu - \cos\theta}$$

$$\Gamma^{\theta}_{\theta\phi} = \frac{1}{2}g^{\theta\theta}g_{\theta\theta,\phi} = \frac{1}{2}\left(\frac{\cosh\mu - \cos\theta}{a}\right)^2 (0) = 0$$

$$\Gamma^{\theta}_{\theta\mu} = \frac{1}{2}g^{\theta\theta}g_{\theta\theta,\mu} = \frac{1}{2}\left(\frac{\cosh\mu - \cos\theta}{a}\right)^2 \left\{\frac{-2a^2\sinh\mu}{(\cosh\mu - \cos\theta)^3}\right\} = \frac{-\sinh\mu}{\cosh\mu - \cos\theta}$$

$$\Gamma^{\theta}_{\phi\phi} = \frac{1}{2}g^{\theta\theta}g_{\phi\phi,\theta} = -\frac{1}{2}\left(\frac{\cosh\mu - \cos\theta}{a}\right)^2 \left\{\frac{-2a^2\sin\theta\cos\theta}{(\cosh\mu - \cos\theta)^2} - \frac{2a^2\sin^3\theta}{(\cosh\mu - \cos\theta)^3}\right\} = -\frac{\sin\theta\cosh\mu - \sin\theta}{\cosh\mu - \cos\theta}$$

$$\Gamma^{\theta}_{\phi\mu} = 0$$

$$\Gamma^{\theta}_{\mu\mu} = \frac{1}{2}g^{\theta\theta}(-g_{\mu\mu,\theta}) = \frac{1}{2}\left(\frac{\cosh\mu - \cos\theta}{a}\right)^2 \left\{\frac{2a^2\sin\theta}{(\cosh\mu - \cos\theta)^3}\right\} = \frac{\sin\theta}{\cosh\mu - \cos\theta}$$

$$\Gamma^{\phi}_{\theta\theta} = \frac{1}{2}g^{\phi\phi}(-g_{\theta\theta,\phi}) = 0$$

$$\Gamma^{\phi}_{\theta\phi} = \frac{1}{2}g^{\phi\phi}g_{\phi\phi,\theta} = \frac{1}{2}\left(\frac{\cosh\mu - \cos\theta}{a\sin\theta}\right)^2 \left\{\frac{2a^2\sin\theta\cos\theta}{(\cosh\mu - \cos\theta)^2} - \frac{2a^2\sin^3\theta}{(\cosh\mu - \cos\theta)^3}\right\} = \frac{\cosh\mu - 1}{\sin\theta(\cosh\mu - \cos\theta)}$$

$$\Gamma^{\phi}_{\theta\mu} = 0$$

$$\Gamma^{\phi}_{\phi\phi} = \frac{1}{2}g^{\phi\phi}g_{\phi\phi,\phi} = 0$$

$$\Gamma^{\phi}_{\phi\mu} = \frac{1}{2}g^{\phi\phi}g_{\phi\phi,\mu} = \frac{1}{2}\left(\frac{\cosh\mu - \cos\theta}{a\sin\theta}\right)^2 \left\{\frac{-2a^2\sin^2\theta\sinh\mu}{(\cosh\mu - \cos\theta)^3}\right\} = -\frac{\sinh\mu}{\cosh\mu - \cos\theta}$$

$$\Gamma^{\phi}_{\mu\mu} = \frac{1}{2}g^{\phi\phi}(-g_{\mu\mu,\phi}) = 0$$

$$\Gamma^{\mu}_{\theta\theta} = \frac{1}{2}g^{\mu\mu}(-g_{\theta\theta,\mu}) = \frac{1}{2}\left(\frac{\cosh\mu - \cos\theta}{a}\right)^2 \left\{\frac{2a^2\sinh\mu}{(\cosh\mu - \cos\theta)^3}\right\} = \frac{\sinh\mu}{\cosh\mu - \cos\theta}$$

$$\Gamma^{\mu}_{\theta\phi} = 0$$

$$\Gamma^{\mu}_{\theta\mu} = \frac{1}{2}g^{\mu\mu}g_{\mu\mu,\theta} = \frac{1}{2}\left(\frac{\cosh\mu - \cos\theta}{a}\right)^2 \left\{\frac{-2a^2\sin\theta}{(\cosh\mu - \cos\theta)^3}\right\} = \frac{\sin\theta}{\cosh\mu - \cos\theta}$$

$$\Gamma^{\mu}_{\phi\phi} = \frac{1}{2} g^{\mu\mu} g_{\phi\phi,\mu} = \frac{1}{2} \left(\frac{\cosh\mu - \cos\theta}{a} \right)^2 \left\{ \frac{2a^2 \sin^2\theta \sinh\mu}{(\cosh\mu - \cos\theta)^3} \right\} = \frac{\sin^2\theta \sinh\mu}{\cosh\mu - \cos\theta}$$

$$\Gamma^{\mu}_{\phi\mu} = \frac{1}{2} g^{\mu\mu} g_{\mu\mu,\phi} = 0$$

$$\Gamma^{\mu}_{\mu\mu} = \frac{1}{2} g^{\mu\mu} g_{\mu\mu,\mu} = \frac{1}{2} \left(\frac{\cosh\mu - \cos\theta}{a} \right)^2 \left\{ \frac{-2a^2 \sinh\mu}{(\cosh\mu - \cos\theta)^3} \right\} = \frac{-\sinh\mu}{\cosh\mu - \cos\theta}$$

$$\begin{aligned}
f^{\theta}_{;\theta} &= f^{\theta}_{,\theta} + f^{\alpha}\Gamma^{\theta}_{\alpha\theta} = f^{\theta}_{,\theta} + f^{\theta}\frac{-\sin\theta}{\cosh\mu - \cos\theta} + f^{\mu}\frac{-\sinh\mu}{\cosh\mu - \cos\theta} \\
f^{\theta}_{;\phi} &= f^{\theta}_{,\phi} + f^{\alpha}\Gamma^{\theta}_{\alpha\phi} = f^{\theta}_{,\phi} + f^{\phi}\frac{\sin\theta - \sin\theta\sinh\mu}{\cosh\mu - \cos\theta} \\
f^{\theta}_{;\mu} &= f^{\theta}_{,\mu} + f^{\alpha}\Gamma^{\theta}_{\alpha\mu} = f^{\theta}_{,mu} + f^{\mu}\frac{\sin\theta}{\cosh\mu - \cos\theta} + f^{\theta}\frac{-\sinh\mu}{\cosh\mu - \cos\theta} \\
f^{\phi}_{;\theta} &= f^{\phi}_{,\theta} + f^{\alpha}\Gamma^{\phi}_{\alpha\theta} = f^{\phi}_{,\theta} + f^{\phi}\frac{\cosh\mu - 1}{\sin\theta(\cosh\mu - \cos\theta)} \\
f^{\phi}_{;\phi} &= f^{\phi}_{,\phi} + f^{\alpha}\Gamma^{\phi}_{\alpha\phi} = f^{\phi}_{,\phi} + f^{\phi}\frac{\cosh\mu - 1}{\sin\theta(\cosh\mu - \cos\theta)} + f^{\mu}\frac{\sinh\mu}{\cosh\mu - \cos\theta} \\
f^{\phi}_{;\mu} &= f^{\phi}_{,\mu} + f^{\alpha}\Gamma^{\phi}_{\alpha\mu} = f^{\phi}_{,\mu} + f^{\phi}\frac{\sinh\mu}{\cosh\mu - \cos\theta} \\
f^{\mu}_{;\theta} &= f^{\mu}_{,\theta} + f^{\alpha}\Gamma^{\mu}_{\alpha\theta} = f^{\mu}_{,\theta} + f^{\mu}\frac{\sinh\mu}{\cosh\mu - \cos\theta} + f^{\mu}\frac{\sin\theta}{\cosh\mu - \cos\theta} \\
f^{\mu}_{;\phi} &= f^{\mu}_{,\phi} + f^{\alpha}\Gamma^{\mu}_{\alpha\phi} = f^{\mu}_{,\phi} + f^{\phi}\frac{\sin^2\theta\sinh\mu}{\cosh\mu - \cos\theta} \\
f^{\mu}_{;\mu} &= f^{\mu}_{,\mu} + f^{\alpha}\Gamma^{\mu}_{\alpha\mu} = f^{\mu}_{,\mu} + f^{\theta}\frac{\sin\theta}{\cosh\mu - \cos\theta} + f^{\mu}\frac{-\sinh\mu}{\cosh\mu - \cos\theta}
\end{aligned}$$

$$x = \lambda\mu\cos\phi \qquad y = \lambda\mu\sin\phi \qquad z = \frac{1}{2}(\lambda^2 - \mu^2)$$

$$\frac{\partial x}{\partial\lambda} = \mu\cos\phi \qquad \frac{\partial y}{\partial\lambda} = \mu\sin\phi \qquad \frac{\partial z}{\partial\lambda} = \lambda$$

$$\frac{\partial x}{\partial\mu} = \lambda\cos\phi \qquad \frac{\partial y}{\partial\mu} = \lambda\sin\phi \qquad \frac{\partial z}{\partial\mu} = -\mu$$

$$\frac{\partial x}{\partial\phi} = -\lambda\mu\sin\phi \qquad \frac{\partial y}{\partial\phi} = \lambda\mu\cos\phi \qquad \frac{\partial z}{\partial\phi} = 0$$

$$\begin{aligned}
g_{\lambda\lambda} &= \left(\frac{\partial x}{\partial\lambda}\right)^2 + \left(\frac{\partial y}{\partial\lambda}\right)^2 + \left(\frac{\partial z}{\partial\lambda}\right)^2 \\
&= \mu^2\cos^2\phi + \mu^2\sin^2\phi + \lambda^2 = \mu^2 + \lambda^2 \\
g_{\mu\mu} &= \left(\frac{\partial x}{\partial\mu}\right)^2 + \left(\frac{\partial y}{\partial\mu}\right)^2 + \left(\frac{\partial x}{\partial\mu}\right)^2 \\
&= \lambda^2\cos^2\phi + \lambda^2\sin^2\phi + \mu^2 = \lambda^2 + \mu^2 \\
g_{\phi\phi} &= \left(\frac{\partial x}{\partial\phi}\right)^2 + \left(\frac{\partial y}{\partial\phi}\right)^2 + \left(\frac{\partial z}{\partial\phi}\right)^2 \\
&= \lambda^2\mu^2\sin^2\phi + \lambda^2\mu^2\cos^2\phi = \lambda^2\mu^2
\end{aligned}$$

$$\Gamma^{\lambda}_{\lambda\lambda} = \frac{1}{2}g^{\lambda\lambda}g_{\lambda\lambda,\lambda} = \frac{1}{2}\left(\frac{1}{\mu^2+\lambda^2}\right)(2\lambda) = \frac{\lambda}{\mu^2+\lambda^2}$$
$$\Gamma^{\lambda}_{\lambda\mu} = \frac{1}{2}g^{\lambda\lambda}g_{\lambda\lambda,\mu} = \frac{1}{2}\left(\frac{1}{\mu^2+\lambda^2}\right)(2\mu) = \frac{\mu}{\lambda^2+\mu^2}$$
$$\Gamma^{\lambda}_{\lambda\phi} = \frac{1}{2}g^{\lambda\lambda}g_{\lambda\lambda,\phi} = 0$$
$$\Gamma^{\lambda}_{\mu\mu} = \frac{1}{2}g^{\lambda\lambda}g_{\mu\mu,\lambda} = \frac{1}{2}\left(\frac{1}{\mu^2+\lambda^2}\right)(-2\lambda) = -\frac{\lambda}{\mu^2+\lambda^2}$$
$$\Gamma^{\lambda}_{\mu\phi} = 0$$
$$\Gamma^{\lambda}_{\phi\phi} = \frac{1}{2}g^{\lambda\lambda}g_{\phi\phi,\lambda} = \frac{1}{2}\left(\frac{1}{\mu^2+\lambda^2}\right)(-2\lambda\mu^2) = -\frac{\lambda\mu^2}{\mu^2+\lambda^2}$$
$$\Gamma^{\mu}_{\lambda\lambda} = \frac{1}{2}g^{\mu\mu}(-g_{\lambda\lambda,\mu}) = \frac{1}{2}\left(\frac{1}{\mu^2+\lambda^2}\right)(-2\lambda\mu^2) = \frac{-\mu}{\mu^2+\lambda^2}$$
$$\Gamma^{\mu}_{\lambda\mu} = \frac{1}{2}g^{\mu\mu}g_{\mu\mu,\lambda} = \frac{1}{2}\left(\frac{1}{\mu^2+\lambda^2}\right)(\lambda) = \frac{\lambda}{\mu^2+\lambda^2} \qquad \Gamma^{\mu}_{\lambda\phi} = 0$$
$$\Gamma^{\mu}_{\mu\mu} = \frac{1}{2}g^{\mu\mu}g_{\mu\mu,\mu} = \frac{1}{2}\left(\frac{1}{\mu^2+\lambda^2}\right)(2\mu) = \frac{\mu}{\mu^2+\lambda^2}$$
$$\Gamma^{\mu}_{\mu\phi} = \frac{1}{2}g^{\mu\mu}g_{\mu\mu,\phi} = 0$$
$$\Gamma^{\mu}_{\phi\phi} = \frac{1}{2}g^{\mu\mu}g_{\phi\phi,\mu} = \frac{1}{2}\left(\frac{1}{\mu^2+\lambda^2}\right)(-2\lambda^2\mu) = \frac{-\lambda^2\mu}{\mu^2+\lambda^2}$$
$$\Gamma^{\phi}_{\lambda\lambda} = \frac{1}{2}g^{\phi\phi}(-g_{\lambda\lambda,\phi}) = 0 \qquad \Gamma^{\phi}_{\lambda\mu} = 0$$
$$\Gamma^{\phi}_{\lambda\phi} = \frac{1}{2}g^{\phi\phi}g_{\phi\phi,\lambda} = \frac{1}{2}\left(\frac{1}{\lambda^2\mu^2}\right)(2\lambda\mu^2) = \frac{1}{\lambda}$$
$$\Gamma^{\phi}_{\mu\mu} = \frac{1}{2}g^{\phi\phi}(-g_{\mu\mu,\phi}) = 0$$
$$\Gamma^{\phi}_{\phi\mu} = \frac{1}{2}g^{\phi\phi}g_{\phi\phi,\mu} = \frac{1}{2}\left(\frac{1}{\lambda^2\mu^2}\right)(\lambda^2\mu) = \frac{1}{\mu}$$
$$\Gamma^{\phi}_{\phi\phi} = \frac{1}{2}g^{\phi\phi}g_{\phi\phi,\phi} = 0$$

$$f^{\lambda}_{;\lambda} = f^{\lambda}_{,\lambda} + f^{\alpha}\Gamma^{\lambda}_{\alpha\lambda} = f^{\lambda}_{,\lambda} + f^{\lambda}\frac{\lambda}{\mu^2+\lambda^2} + f^{\mu}\frac{\mu}{\mu^2+\lambda^2}$$
$$f^{\lambda}_{;\phi} = f^{\lambda}_{,\phi} + f^{\alpha}\Gamma^{\lambda}_{\alpha\phi} = f^{\lambda}_{,\phi} + f^{\phi}\frac{-\lambda\mu^2}{\mu^2+\lambda^2}$$
$$f^{\lambda}_{;\mu} = f^{\lambda}_{,\mu} + f^{\alpha}\Gamma^{\lambda}_{\alpha\mu} = f^{\lambda}_{,mu} + f^{\mu}\frac{-\lambda}{\mu^2+\lambda^2} + f^{\lambda}\frac{\mu}{\mu^2+\lambda^2}$$
$$f^{\phi}_{;\lambda} = f^{\phi}_{,\lambda} + f^{\alpha}\Gamma^{\phi}_{\alpha\lambda} = f^{\phi}_{,\lambda} + f^{\phi}\frac{1}{\lambda}$$
$$f^{\phi}_{;\phi} = f^{\phi}_{,\phi} + f^{\alpha}\Gamma^{\phi}_{\alpha\phi} = f^{\phi}_{,\phi} + f^{\lambda}\frac{1}{\lambda} + f^{\mu}\frac{1}{\mu}$$
$$f^{\phi}_{;\mu} = f^{\phi}_{,\mu} + f^{\alpha}\Gamma^{\phi}_{\alpha\mu} = f^{\phi}_{,\mu} + f^{\phi}\frac{1}{\mu}$$
$$f^{\mu}_{;\lambda} = f^{\mu}_{,\lambda} + f^{\alpha}\Gamma^{\mu}_{\alpha\lambda} = f^{\mu}_{,\lambda} + f^{\lambda}\frac{-\mu}{\mu^2+\lambda^2} + f^{\mu}\frac{\lambda}{\mu^2+\lambda^2}$$
$$f^{\mu}_{;\phi} = f^{\mu}_{,\phi} + f^{\alpha}\Gamma^{\mu}_{\alpha\phi} = f^{\mu}_{,\phi} + f^{\phi}\frac{-\lambda^2\mu}{\mu^2+\lambda^2}$$
$$f^{\mu}_{;\mu} = f^{\mu}_{,\mu} + f^{\alpha}\Gamma^{\mu}_{\alpha\mu} = f^{\mu}_{,\mu} + f^{\lambda}\frac{\lambda}{\mu^2+\lambda^2} + f^{\mu}\frac{\mu}{\mu^2+\lambda^2}$$

Problem 1.23

$$\frac{d^2}{dt^2}\begin{pmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{pmatrix} + \begin{pmatrix} \omega^2 & -k^2 & 0 \\ -k^2 & \omega^2 & -k^2 \\ 0 & -k^2 & \omega^2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0$$

$$\begin{vmatrix} \omega^2-\lambda & -k^2 & 0 \\ -k^2 & \omega^2-\lambda & -k^2 \\ 0 & -k^2 & \omega^2-\lambda \end{vmatrix} = (\omega^2-\lambda)\left\{(\omega^2-\lambda)^2-k^4\right\} - k^2\left\{k^2(\omega^2-\lambda)\right\} = 0$$

$$(\omega^2-\lambda)\left\{(\omega^2-\lambda)^2-2k^4\right\} = 0$$

$$\lambda_1 = \omega^2 \qquad \lambda_{2,3} = \omega^2 \pm k^2\sqrt{2}$$

$$\lambda_1 = \omega^2 \qquad \begin{aligned} 0\mathbf{e}_1 \; -k^2\mathbf{e}_2 \; +0\mathbf{e}_3 &= 0 \\ -k^2\mathbf{e}_1 \; +0\mathbf{e}_2 \; -k^2\mathbf{e}_3 &= 0 \\ 0\mathbf{e}_1 \; -k^2\mathbf{e}_2 \; +0\mathbf{e}_3 &= 0 \\ \mathbf{e}_2 &= 0 \\ \mathbf{e}_3 &= -\mathbf{e}_1 \end{aligned} \qquad \mathbf{e}_{\omega^2} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\lambda_{2,3} = \omega^2 \pm k^2\sqrt{2} \qquad \begin{aligned} \mp k^2\sqrt{2}\mathbf{e}_1 \; -k^2\mathbf{e}_2 \; +0\mathbf{e}_3 &= 0 \\ -k^2\mathbf{e}_1 \; \mp k^2\sqrt{2}\mathbf{e}_2 \; -k^2\mathbf{e}_3 &= 0 \\ 0\mathbf{e}_1 \; -k^2\mathbf{e}_2 \; +0\mathbf{e}_3 &= 0 \\ \mathbf{e}_2 &= \mp\sqrt{2}\mathbf{e}_1 \\ \mathbf{e}_2 &= \mp\sqrt{2}\mathbf{e}_3 \\ \mathbf{e}_3 &= \mathbf{e}_1 \end{aligned} \qquad \mathbf{e}_{\omega^2 \pm k^2\sqrt{2}} = \begin{pmatrix} \frac{1}{2} \\ \mp\frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{pmatrix}$$

Problem 1.31

$$T_{\mu\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$$T'_{\mu\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T'_{\mu\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\rho\gamma & p\beta\gamma & 0 & 0 \\ -p\beta\gamma & p\gamma & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$$T'_{\mu\nu} = \begin{pmatrix} -\rho\gamma^2 + p\beta^2\gamma^2 & \rho\beta\gamma^2 - p\beta\gamma^2 & 0 & 0 \\ \rho\beta\gamma^2 - p\beta\gamma^2 & -\rho\beta^2\gamma^2 + p\gamma^2 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$$\beta = 0.8 \qquad \gamma = \frac{1}{\sqrt{1 - 0.8^2}} = \frac{1}{0.6} = 1.67$$

$$\beta^2 = 0.64 \qquad \gamma^2 = 2.78$$

$$T'_{\mu\nu} = \begin{pmatrix} -2.78\rho + 1.78p & 1.78\rho - 1.78p & 0 & 0 \\ 1.78\rho - 1.78p & 1.78\rho - 2.78p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Problem 1.32

For a Lorentz boost along the x-direction,

$$\mathbf{S}' = \begin{pmatrix} e^{-\frac{\alpha}{2}} & 0 \\ 0 & e^{\frac{\alpha}{2}} \end{pmatrix} \mathbf{S}$$

For a space rotation $(\phi\theta\psi)$

$$\mathbf{S} = \begin{pmatrix} \cos\frac{\theta}{2} e^{-\frac{i}{2}(\psi+\phi)} & \sin\frac{\theta}{2} e^{-\frac{i}{2}(\psi-\phi)} \\ -\sin\frac{\theta}{2} e^{\frac{i}{2}(\psi-\phi)} & \cos\frac{\theta}{2} e^{\frac{i}{2}(\psi+\phi)} \end{pmatrix} \mathbf{S}''$$

For the space rotation followed by the Lorentz boost,

$$\mathbf{S}' = \begin{pmatrix} e^{-\frac{\alpha}{2}} & 0 \\ 0 & e^{\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} e^{-\frac{i}{2}(\psi+\phi)} & \sin\frac{\theta}{2} e^{-\frac{i}{2}(\psi-\phi)} \\ -\sin\frac{\theta}{2} e^{\frac{i}{2}(\psi-\phi)} & \cos\frac{\theta}{2} e^{\frac{i}{2}(\psi+\phi)} \end{pmatrix} \mathbf{S}''$$

$$\mathbf{S}' = \mathbf{S} = \begin{pmatrix} \cos\frac{\theta}{2} e^{-\frac{1}{2}(\alpha+i\phi+i\psi)} & \sin\frac{\theta}{2} e^{-\frac{1}{2}(\alpha+i\psi-i\phi)} \\ -\sin\frac{\theta}{2} e^{\frac{1}{2}(-\alpha+i\psi-i\phi)} & \cos\frac{\theta}{2} e^{\frac{1}{2}(\alpha+i\psi+i\phi)} \end{pmatrix} \mathbf{S}''$$

Problem 2.1

Assumptions: adiabatic gas behavior
constant tension membrane
constant density membrane

$$pV^\gamma = p_0 V_0^\gamma$$

$$p = p_0 V_0^\gamma V^{-\gamma}$$

$$\Delta p = -\gamma p_0 V_0^\gamma V^{-\gamma-1} \Delta V$$

$$\Delta p = -\gamma p_0 \frac{V_0^\gamma}{V^\gamma} \frac{\Delta V}{V}$$

$$\Delta V = \int \psi \, dA$$

$$\Delta p = -\frac{\gamma p}{V} \int \psi \, dA$$

Speed of sound under adiabatic conditions: $c^2 = \dfrac{\gamma p}{\rho}$

$$\Delta p = -\frac{\rho c^2}{V} \int \psi \, dA$$

Newton's second law for a differential of area:

$$\sigma \Delta x \Delta y \frac{\partial^2 \psi}{\partial t^2} = T \Delta x \Delta \frac{\partial \psi}{\partial y} + T \Delta y \Delta \frac{\partial \psi}{\partial x} + \Delta p \Delta x \Delta y$$

$$\sigma \Delta x \Delta y \frac{\partial^2 \psi}{\partial t^2} = T \Delta x \Delta y \frac{\partial^2 \psi}{\partial y^2} + T \Delta y \Delta x \frac{\partial^2 \psi}{\partial x^2} - \frac{\rho c^2}{V} \Delta x \Delta y \int \psi \, dA$$

$$\frac{\sigma}{T} \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\rho c^2}{VT} \int \psi \, dA$$

$$v^2 = \left(\frac{\sigma}{T}\right)^{-1}$$

$$\frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\rho c^2}{VT} \int \psi \, dA$$

Problem 2.4

The mechanical equation is

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \frac{1}{c}\mathbf{J}\times\mathbf{B}$$

$$\mathbf{J} = \sigma\mathbf{E}$$

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \frac{\sigma}{c}\mathbf{E}\times\mathbf{B}$$

$$\nabla\times\mathbf{H} = \frac{1}{c}\left(\frac{\partial \mathbf{D}}{\partial t} + 4\pi\mathbf{J}\right) = \frac{1}{c}\left(\epsilon\frac{\partial \mathbf{E}}{\partial t} + 4\pi\sigma\mathbf{E}\right)$$

$$\nabla\times\mathbf{B} = \frac{\mu}{c}\left(\frac{\partial \mathbf{D}}{\partial t} + 4\pi\mathbf{J}\right)$$

Inside a conductor,

$$\mathbf{E} = -4\pi\mathbf{P} \qquad \mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = 0 \qquad \mathbf{D} = \epsilon\mathbf{E} \qquad \epsilon = 0$$

$$\nabla\times\mathbf{D} = \frac{4\pi\mu\sigma}{c}\mathbf{E}$$

$$\mathbf{E} = \frac{c}{4\pi\mu\sigma}\nabla\times\mathbf{B}$$

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \frac{\sigma}{c}\mathbf{B}\times\frac{c}{4\pi\mu\sigma}\nabla\times\mathbf{B}$$

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \frac{1}{4\pi\mu}\mathbf{B}\times\nabla\times\mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla\times\mathbf{E}'$$

$\mathbf{E}'$ here is the field as seen by a stationary observer. The relationship $\mathbf{J} = \sigma\mathbf{E}$ is valid as seen by an observer moving with the fluid, obtained through diffusion analysis on the force equation $\mathbf{F} = \rho\mathbf{E}$. In the moving frame, $\mathbf{F} = \rho\mathbf{E}' + \frac{\rho}{c}\mathbf{v}\times\mathbf{B}$, so that

$$\mathbf{E}' = \frac{1}{\sigma}\mathbf{J} - \frac{1}{c}\mathbf{v}\times\mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla\times\left\{\frac{1}{\sigma}\mathbf{J} - \frac{1}{c}\mathbf{v}\times\mathbf{B}\right\}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla\times\left\{\frac{c}{4\pi\mu\sigma}\nabla\times\mathbf{B} - \frac{1}{c}\mathbf{v}\times\mathbf{B}\right\}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) - \frac{c^2}{4\pi\mu\sigma}\nabla\times\nabla\times\mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \frac{c^2}{4\pi\mu\sigma}\nabla^2\mathbf{B}$$

Problem 2.5

From the continuity equation,

$$\frac{\partial c_1}{\partial t} = -\nabla \cdot \mathbf{J}$$

$\mathbf{J}$ = drift flux + concentration gradient flux

$$\frac{\partial c_1}{\partial t} = -c_1\mathbf{v} - A_1^2\nabla c_1$$

$$\mathbf{v} = B_1\mathbf{F} = B_1Q\mathbf{E} = -B_1Q\nabla\phi$$

$$\mathbf{J} = -c_1B_1Q\nabla\phi - A_1^2\nabla c_1$$

$$\frac{\partial c_1}{\partial t} = -\nabla \cdot \mathbf{J} = A_1^2\nabla^2 c_1 + B_1Q\nabla \cdot (c_1\nabla\phi)$$

Problem 3.1

(a.)

$$p\dot{q} - H - P\dot{Q} + K = \dot{S} \qquad S = S(q, Q, t)$$

$$p\dot{q} - H - \frac{d}{dt}(QP) + Q\dot{P} + K = \dot{S}$$

$$p\dot{q} - H + Q\dot{P} + K = \dot{S} + \frac{d}{dt}(QP)$$

$$p\dot{q} - H + Q\dot{P} + K = \dot{S}' \qquad S' = S + QP$$

$$p\,dq + Q\,dP + (K - H)\,dt = dS'$$

$$p\,dq + Q\,dP + (K - H)\,dt = \frac{\partial S'}{\partial q}\,dq + \frac{\partial S'}{\partial P} + \frac{\partial S'}{\partial t}$$

$$p = \frac{\partial S'}{\partial q} \qquad Q = \frac{\partial S'}{\partial P} \qquad K = H + \frac{\partial S'}{\partial t}$$

(b.)

$$S' = qP$$

$$p = \frac{\partial S'}{\partial q} = P \qquad Q = \frac{\partial S'}{\partial P} = q \qquad \frac{\partial S'}{\partial t} = 0$$

$$K = H + \frac{\partial S'}{\partial t} = H \qquad Q = q \qquad P = p$$

The new coordinate and new momentum are identical with the old.

(c.)

$$S' = qP + \epsilon T(q, P) = qP + \epsilon T(q, p) \qquad \epsilon << 1$$

$$p = \frac{\partial S'}{\partial q} = P + \epsilon \frac{\partial T}{\partial q} \qquad Q = \frac{\partial S'}{\partial P} = q + \epsilon \frac{\partial T}{\partial P}$$

$$P - p = P - \left(P + \epsilon \frac{\partial T}{\partial q}\right) = -\epsilon \frac{\partial T}{\partial q} \qquad Q - q = q + \epsilon \frac{\partial T}{\partial P} - q = \epsilon \frac{\partial T}{\partial P}$$

(d.)

$$\Delta f = f(P, Q) - f(p, q)$$

$$\Delta f = f(p, q) + \frac{\partial f}{\partial p}\,(P - p) + \frac{\partial f}{\partial q}\,(Q - q) - f(p, q)$$

$$\Delta f = \frac{\partial f}{\partial p}\left(-\epsilon\frac{\partial T}{\partial q}\right) + \frac{\partial f}{\partial q}\left(\epsilon\frac{\partial T}{\partial P}\right) = \epsilon\left\{\frac{\partial f}{\partial q}\frac{\partial T}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial T}{\partial q}\right\}$$

$$\Delta f = \epsilon[f, T]$$

Apply this to the Hamiltonian

$$\Delta H = \epsilon[H, T]$$

If the Hamiltonian is invariant,

$$\Delta H = 0 = \epsilon[H, T]$$

Since ϵ is arbitrary,

$$0 = [H, T]$$

and T is a constant of the motion.

(e.)

$$\mathbf{R} - \mathbf{r} = \epsilon\times\mathbf{r} = (-\epsilon y, \epsilon x, 0)$$

$$\mathbf{P} - \mathbf{p} = \epsilon\times\mathbf{p} = (-\epsilon p_y, \epsilon p_x, 0)$$

$$X - x = -\epsilon y = \epsilon\frac{\partial T}{\partial p_x}$$

$$Y - y = \epsilon x = \epsilon\frac{\partial T}{\partial p_y} \qquad T = -yp_x + xp_y + f(x, y, z)$$

$$Z - z = 0 = \epsilon\frac{\partial T}{\partial p_z}$$

$$P_x - p_x = -\epsilon p_y = -\epsilon\frac{\partial T}{\partial x}$$

$$P_y - p_y = \epsilon p_x = -\epsilon\frac{\partial T}{\partial y} \qquad T = xp_y - yp_x + g(p_x, p_y, p_z)$$

$$P_z - p_z = 0 = -\epsilon\frac{\partial T}{\partial z}$$

$$T = xp_y - yp_x = (\mathbf{r}\times\mathbf{p})_z = \mathbf{M}_z$$

Problem 3.2

$$\int_{t_1}^{t_2} L'\,dt = \int_{t_1}^{t_2} \left(L + \frac{df}{dt}\right) = \int_{t_1}^{t_2} L\,dt + \int_{t_1}^{t_2} \frac{dF}{dt}\,dt$$

$$\int_{t_1}^{t_2} L'\,dt = \int_{t_1}^{t_2} L\,dt + \int_{F_1}^{F_2} dF$$

$$\int_{t_1}^{t_2} L'\,dt = \int_{t_1}^{t_2} L\,dt + F(t_2) - F(t_1)$$

$$\delta \int_{t_1}^{t_2} L'\,dt = \delta \int_{t_1}^{t_2} L\,dt + \delta[(F(t_2) - F(t_1)]$$

$$\delta \int_{t_1}^{t_2} L'\,dt = \delta \int_{t_1}^{t_2} L\,dt$$

$\delta[F(t_2) - F(t_1)] = 0$ because no variation is allowed at the endpoints. Minimizing both sides of the last equation yields

$$\frac{\partial L'}{\partial t} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{x}}\right) = 0 \qquad \frac{\partial L}{\partial t} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

and the form of the Lagrange equations is unchanged.

For a non-relativistic particle in an electromagnetic field,

$$T = \frac{1}{2}mv^2$$

$$V = e\phi - \frac{e}{c}(\mathbf{v} \cdot \mathbf{A})$$

and the simplest Lagrangian is

$$L = T - V = \frac{1}{2}mv^2 - e\phi + \frac{e}{c}(\mathbf{v} \cdot \mathbf{A})$$

$$L = T - V = \frac{1}{2}mv^2 - e\phi + \frac{e}{c}\left(\frac{d\mathbf{r}}{dt} \cdot \mathbf{A}\right)$$

$$L = T - V = \frac{1}{2}mv^2 - e\phi + \frac{e}{c}\left(\frac{d}{dt}\mathbf{r} \cdot \mathbf{A} - \mathbf{r} \cdot \frac{d\mathbf{A}}{dt}\right)$$

$$L = T - V = \frac{1}{2}mv^2 - e\phi - \frac{e}{c}\mathbf{r} \cdot \frac{d\mathbf{A}}{dt} + \frac{e}{c}\frac{d}{dt}\mathbf{r} \cdot \mathbf{A}$$

The last term may be discarded as a total time derivative.

$$L = T - V = \frac{1}{2}mv^2 - e\phi - \frac{e}{c}\mathbf{r} \cdot \left\{\frac{\partial \mathbf{A}}{\partial \mathbf{r}}\frac{d\mathbf{r}}{dt} + \frac{\partial \mathbf{A}}{\partial t}\right\}$$

$$L = T - V = \frac{1}{2}mv^2 - e\phi - \frac{e}{c}\mathbf{r} \cdot \left\{ \nabla\mathbf{A} \cdot \mathbf{v} + \frac{\partial \mathbf{A}}{\partial t} \right\}$$

$$\frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} - \frac{e}{c}\mathbf{r} \cdot \nabla\mathbf{A} = \mathbf{p}$$

$$H = \mathbf{p} \cdot \mathbf{r} - L$$

$$H = \mathbf{p} \cdot \mathbf{r} - \left\{ \frac{1}{2}m \left(\frac{\mathbf{p} + \frac{e}{c}\mathbf{r} \cdot \nabla\mathbf{A}}{m} \right)^2 - e\phi - \frac{e}{c}\mathbf{r} \cdot \left[\nabla\mathbf{A} \cdot \left(\frac{\mathbf{p} + \frac{e}{c}\mathbf{r} \cdot \nabla\mathbf{A}}{m} \right) + \frac{\partial \mathbf{A}}{\partial t} \right] \right\}$$

$$H = \mathbf{p} \cdot \mathbf{r} - \frac{1}{2m} \left(\mathbf{p} + \frac{e}{c}\mathbf{r} \cdot \nabla\mathbf{A} \right)^2 + e\phi + \frac{e}{c}\mathbf{r} \cdot \left\{ \nabla\mathbf{A} \cdot \left(\frac{\mathbf{p} + \frac{e}{c}\mathbf{r} \cdot \nabla\mathbf{A}}{m} \right) \right\} + \frac{e}{c}\mathbf{r} \cdot \frac{\partial \mathbf{A}}{\partial t}$$

$$H = \mathbf{p} \cdot \left(\frac{\mathbf{p} + \frac{e}{c}\mathbf{r} \cdot \nabla\mathbf{A}}{m} \right) - \frac{1}{2m}\mathbf{p} \cdot \left(\mathbf{p} + \frac{e}{c}\mathbf{r} \cdot \nabla\mathbf{A} \right) - \frac{1}{2m} \left(\frac{e}{c}\mathbf{r} \cdot \nabla\mathbf{A} \right) \cdot \left(\mathbf{p} + \frac{e}{c}\mathbf{r} \cdot \nabla\mathbf{A} \right) + e\phi$$

$$+ \frac{1}{m} \left(\frac{e}{c}\mathbf{r} \cdot \nabla\mathbf{A} \right) \cdot \left(\mathbf{p} + \frac{e}{c}\mathbf{r} \cdot \nabla\mathbf{A} \right) + \frac{e}{c}\mathbf{r} \cdot \frac{\partial \mathbf{A}}{\partial t}$$

$$H = \frac{\mathbf{p}}{2m} \cdot \left(\mathbf{p} + \frac{e}{c}\mathbf{r} \cdot \nabla\mathbf{A} \right) + \frac{1}{2m} \left(\frac{e}{c}\mathbf{r} \cdot \nabla\mathbf{A} \right) \cdot \left(\mathbf{p} + \frac{e}{c}\mathbf{r} \cdot \nabla\mathbf{A} \right) + e\phi + \frac{e}{c}\mathbf{r} \cdot \frac{\partial \mathbf{A}}{\partial t}$$

$$H = \frac{1}{2m} \left| \mathbf{p} + \frac{e}{c}\mathbf{r} \cdot \nabla\mathbf{A} \right|^2 + e\phi + \frac{e}{c}\mathbf{r} \cdot \frac{\partial \mathbf{A}}{\partial t}$$

Problem 3.3

$$I = \int_V \int_t L\sqrt{g}\, d\xi_1\, d\xi_2\, d\xi_3\, dt$$

The Lagrange-Euler equations are applied to $L\sqrt{g}$:

$$\frac{\partial}{\partial\psi}(L\sqrt{g}) - \frac{\partial}{\partial\xi^\alpha}\frac{\partial}{\partial\psi_{,\alpha}}(L\sqrt{g}) - \frac{\partial}{\partial t}\frac{\partial}{\partial\dot{\psi}}(L\sqrt{g}) = 0$$

$$\sqrt{g}\,\frac{\partial L}{\partial\psi} - \frac{\partial}{\partial\xi^\alpha}\sqrt{g}\,\frac{\partial L}{\partial\psi_{,\alpha}} - \sqrt{g}\,\frac{\partial L}{\partial t}\,\frac{\partial}{\partial\dot{\psi}} = 0$$

$$\frac{\partial L}{\partial\psi} - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi^\alpha}\sqrt{g}\,\frac{\partial L}{\partial\psi_{,\alpha}} - \frac{\partial L}{\partial t}\,\frac{\partial}{\partial\dot{\psi}} = 0$$

If $L = (\nabla\psi)^2$, then in cartesian coordinates,

$$L = \delta^{ij}\psi_{,i}\psi_{,j}$$

$$\frac{\partial L}{\partial\psi} = 0 \qquad \frac{\partial L}{\partial\psi_{,}{}^{i}} = 2\delta^{ij}\psi_{,j} \qquad \frac{\partial L}{\partial\dot{\psi}} = 0 \qquad \sqrt{g} = 1$$

$$-\frac{\partial L}{\partial x^i}\,(2\delta^{ij}\psi_{,j}) = 0$$

$$-2\delta^{ij}\psi_{,i}\psi_{,j} = 0$$

$$-2\nabla^2\psi = 0$$

Since this in an invariant statement,

$$L = g^{ij}\psi_{,i}\psi_{,j}$$

$$\frac{\partial L}{\partial\psi} = 0 \qquad \frac{\partial L}{\partial\psi_{,i}} = 2g^{ij}\psi_{,j} \qquad \frac{\partial L}{\partial\dot{\psi}} = 0$$

$$-\frac{1}{\sqrt{g}}\,\frac{\partial}{\partial\xi^i}\,\sqrt{g}\,(2g^{ij}\psi_{,i}) = -2\nabla^2\psi = 0$$

$$\sqrt{g} = h_1h_2h_3$$

$$g^{ij} = \prod_{i=1}^{3}(h_i)^{-2}$$

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\sum_{i=1}^{3}\frac{\partial}{\partial\xi^i}\,\frac{h_1h_2h_3}{\prod_{i=1}^{3}h_i^2}\,\frac{\partial}{\partial\xi^i}$$

Problem 3.4

$$M^{\mu\nu\lambda} = T^{\mu\nu}x^{\lambda} - T^{\mu\lambda}x^{\nu}$$

$$M^{\mu\nu\lambda}{}_{;\mu} = T^{\mu\nu}{}_{;\mu}x^{\lambda} + T^{\mu\nu}x^{\lambda}{}_{;\mu} - T^{\mu\lambda}{}_{;\mu}x^{\nu} - T^{\mu\lambda}{}_{;\mu}x^{\nu} - T^{\mu\lambda}x^{\nu}{}_{;\mu}$$

$$M^{\mu\nu\lambda}{}_{;\mu} = T^{\mu\nu}{}_{;\mu}x^{\lambda} - T^{\mu\lambda}{}_{;\mu}x^{\nu} + T^{\lambda\nu} - T^{\nu\lambda}$$

$$T^{\mu\nu}{}_{;\mu} = 0$$

$$M^{\mu\nu\lambda}{}_{;\mu} = T^{\lambda\nu} - T^{\nu\lambda}$$

$$T^{\mu\lambda} = T^{\lambda\mu}$$

$$M^{\mu\nu\lambda}{}_{;\mu} = 0$$

$$M^{4\nu\lambda}{}_{;4} = -M^{i\nu\lambda}{}_{;i}$$

$$T^{\mu\lambda} = T^{\lambda\mu}$$

$$M^{4\mu\nu} = T^{4\mu}x^{\nu} - T^{4\nu}x^{\mu}$$

$$M^{4ij} = T^{4i}x^{j} - T^{4j}x^{i}$$

$$T^{4i} = P^{i} = \mathbf{i} \text{ component of momentum density}$$

$$M^{4ij} = P^{i}x^{j} - P^{j}x^{i} = \mathbf{k} \text{ component of } -\mathbf{P}\times\mathbf{R}$$

$$M^{4ij} = (\mathbf{R}\times\mathbf{P})_{k} \qquad i \neq j \neq k$$

$$M^{4ij} = \text{angular momentum density}$$

$$H = \text{total angular momentum}$$

$$H = \int dV\, M^{4iv}$$

$$\frac{dH}{dt} = \int dV\, \frac{\partial}{\partial t} M^{4ij} = \int dV \left\{ -\frac{\partial M^{kij}}{\partial x^{k}} \right\} = -\int dV\, \nabla \cdot M^{kij}$$

$$\frac{dH}{dt} = -\int dA\, \hat{n}_{k} \cdot M^{kij} = -\int dA\, \hat{n}_{k} \left\{ T^{ki}x^{j} - T^{kj}x^{i} \right\}$$

If the system is closed at a finite distance, $T^{ij} = 0$ there and $\dfrac{dH}{dt} = 0$ or, total angular momentum is constant.

Problem 4.1

$$\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a + b\cos\theta} = \oint \frac{\left(\frac{z-z^*}{2i}\right)^2}{a + b\left(\frac{z+z^*}{2}\right)} \frac{dz}{iz}$$

$$\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a + b\cos\theta} = \oint \frac{z^2 - 2zz^* + (z^*)^2}{(-2i)(bz^2 + 2az + bzz^*)}\, dz$$

$$\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a + b\cos\theta} = \oint \frac{z^2 + (z^*)^2 - 2}{-2i(bz^2 + 2az + b)}\, dz$$

$$\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a + b\cos\theta} = \oint \frac{z^2 + (z^*)^2 - 2}{-2ib\left(z + \frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1}\right)\left(z + \frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1}\right)}\, dz$$

$|z| = \left|-\frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1}\right| = \frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1} > 1$ if $a > b$ so that the only pole inside the contour is $z = -\frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1}$.

$$\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a + b\cos\theta} = 2\pi i \left.\frac{z^2 + (z^*)^2 - 2}{(-2ib)\left(z + \frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1}\right)}\right|_{z=-\frac{a}{b}-\sqrt{\frac{a^2}{b^2}-1}}$$

$$\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a + b\cos\theta} = -\frac{\pi}{b} \frac{2\left(-\frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1}\right)^2 - 2}{-\frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1} + \frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1}}$$

$$\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a + b\cos\theta} = -\frac{2\pi}{b} \frac{\frac{a^2}{b^2} - \frac{a}{b}\sqrt{\frac{a^2}{b^2} - 1} - 1}{\sqrt{\frac{a^2}{b^2} - 1}}$$

$$\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a + b\cos\theta} = -\frac{2\pi}{b}\sqrt{\frac{a^2}{b^2} - 1} + \frac{2a\pi}{b^2} = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right)$$

Problem 4.3

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = \int_{-\infty}^{\infty} \frac{1}{2} \frac{e^{ix}+e^{-ix}}{(x^2+a^2)(x^2+b^2)}\, dx$$

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = \int_{-\infty}^{\infty} \frac{1}{2} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\, dx + \int_{-\infty}^{\infty} \frac{1}{2} \frac{e^{-ix}}{(x^2+a^2)(x^2+b^2)}\, dx$$

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = \int_{-\infty}^{\infty} \frac{1}{2} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\, dx + \int_{-(-\infty)}^{-\infty} \frac{1}{2} \frac{e^{-i(-u)}}{((-u)^2+a^2)((-u)^2+b^2)}\, d(-u)$$

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = \int_{-\infty}^{\infty} \frac{1}{2} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\, dx + \int_{\infty}^{-\infty} \frac{1}{2} \frac{-e^{iu}}{(u^2+a^2)(u^2+b^2)}\, du$$

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = \int_{-\infty}^{\infty} \frac{1}{2} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\, dx + \int_{-\infty}^{\infty} \frac{1}{2} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\, dx$$

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\, dx$$

Close the contour with an infinite half-circle in the upper half-plane, whose contribution to the integral is zero.

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = \oint \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\, dz = \oint \frac{e^{iz}\, dz}{(z-ia)(z+ia)(z-ib)(z+ib)}$$

$$\int_{-\infty}^{\infty} \frac{1}{2} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i \left\{ f_1(ia) + f_2(ib) \right\}$$

$$f_1 = \frac{e^{iz}}{(z+ia)(z-ib)(z+ib)}$$

$$f_2 = \frac{e^{iz}}{(z+ia)(z-ia)(z+ib)}$$

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i \left\{ \frac{e^{-ia}}{2ia(b^2-a^2)} + \frac{e^{-ib}}{2ib(a^2-b^2)} \right\}$$

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2} \left\{ \frac{e^{-b}}{b} + \frac{e^{-a}}{a} \right\}$$

For $a = b$, Re $a > 0$,

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)^2} = \oint \frac{e^{iz}\, dz}{(z^2+a^2)^2} = \oint \frac{e^{iz}\, dz}{(z-ia)^2(z+ia)^2}$$

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i f'(ia)$$

$$f(z) = \frac{e^{iz}}{(z+ia)^2}$$

$$f'(z) = \frac{ie^{iz}(z+ia) - 2e^{iz}}{(z+ia)^3}$$

$$f'(ia) = \frac{ie^{-a}(2ia) - 2e^{-a}}{(2ia)^3} = \frac{e^{-a}(a+1)}{4a^3 i}$$

$$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2+a^2)^2} = \frac{\pi(a+1)e^{-a}}{2a^3 i}$$

Problem 4.4

$$\int_0^{2\pi} e^{\cos\theta}\cos(n\theta-\sin\theta)\,d\theta = \int_0^{2\pi} e^{\cos\theta}\,\frac{1}{2}\,\left(e^{in\theta-i\sin\theta}+e^{-in\theta+i\sin\theta}\right)\,d\theta$$

$$\int_0^{2\pi} e^{\cos\theta}\cos(n\theta-\sin\theta)\,d\theta = \int_0^{2\pi} \frac{1}{2}\,\left(e^{in\theta}e^{\cos\theta-i\sin\theta}+e^{-in\theta}e^{\cos\theta+i\sin\theta}\right)\,d\theta$$

$$\int_0^{2\pi} e^{\cos\theta}\cos(n\theta-\sin\theta)\,d\theta = \oint \frac{1}{2}\,\frac{e^{z^*}z^n+e^z(z^*)^n}{iz}\,dz$$

$$\left(z=e^{i\theta}\right)$$

$$f(z)=\frac{1}{2i}\left(e^{z^*}z^n+e^z(z^*)^n\right)$$

$$f(z)=\frac{1}{2i}\left(z^n\{1+z^*+\frac{1}{2}(z^*)^2+\ldots\}+(z^*)\{1+z+\frac{1}{2}z^2+\ldots\}\right)$$

$$f(z)=\frac{\left(\left\{z^n+z^{n-1}+\frac{1}{2}z^{n-2}+\ldots+\frac{1}{n!}+\frac{z^*}{(n+1)!}+\ldots\right\}+\left\{(z^*)^n+(z^*)^{n-1}+\frac{1}{2}(z^*)^{n-2}+\ldots+\frac{1}{n!}+\frac{z}{(n+1)!}+\ldots\right\}\right)}{2i}$$

$$f(0)=\frac{1}{2i}\left(\frac{1}{n!}+\frac{1}{n!}\right)=\frac{1}{in!}$$

$$\int_0^{2\pi} e^{\cos\theta}\cos(n\theta-\sin\theta)\,d\theta = \oint \frac{1}{2}\,\frac{e^{z^*}z^n+e^z(z^*)^n}{iz}\,dz = 2\pi i f(0)$$

$$\int_0^{2\pi} e^{\cos\theta}\cos(n\theta-\sin\theta)\,d\theta = 2\pi i\left(\frac{1}{in!}\right)=\frac{2\pi}{n!}$$

Problem 4.5

$$\oint \frac{z^{2a-1}}{(z-i)(z+i)}\,dz = \lim_{\rho\to\infty}\int_0^{\pi} \frac{(\rho e^{i\theta})^{(2a-1)}}{(\rho e^{i\theta}-i)(\rho e^{i\theta}+1)}\, ie^{i\theta}\,d\theta + \int_{-\infty}^{\infty} \frac{x^{2a-1}}{(x^2+1)}\,dx$$

The only pole within the contour is $z=i$. The line integral along the half-circle at $\rho\to\infty$ is zero.

$$\oint \frac{z^{2a-1}}{(z-i)(z+i)}\,dz = 2\pi f(i) = \int_{-\infty}^{\infty} \frac{x^{2a-1}}{x^2+1}\,dx$$

$$2\pi\left\{\frac{i^{2a-1}}{2i}\right\} = \int_{-\infty}^{0} \frac{x^{2a-1}}{x^2+1}\,dx + \int_0^{\infty} \frac{x^{2a-1}}{x^2+1}\,dx$$

$$\pi(-1)^a(i)^{-1} = \int_{-(-\infty)}^{0} \frac{(-x)^{2a-1}}{(-x)^2+1}\,d(-x) + \int_0^{\infty} \frac{x^{2a-1}}{x^2+1}\,dx$$

$$-i\pi(-1)^a = \int_0^{\infty} \frac{(-1)^{2a-1}x^{2a-1}}{x^2+1}\,dx + \int_0^{\infty} \frac{x^{2a-1}}{x^2+1}\,dx$$

$$-i\pi(-1)^a = \left(1-(-1)^{2a}\right)\int_0^{\infty} \frac{x^{2a-1}}{x^2+1}\,dx$$

$$-i\pi e^{i\pi a} = \left(1-e^{2i\pi a}\right)\int_0^{\infty} \frac{x^{2a-1}}{x^2+1}\,dx$$

$$\int_0^{\infty} \frac{x^{2a-1}}{x^2+1}\,dx = \frac{-i\pi e^{i\pi a}}{1-e^{i2\pi a}} = \frac{-i\pi}{e^{-i\pi a}-e^{i\pi a}}$$

$$\int_0^{\infty} \frac{x^{2a-1}}{x^2+1}\,dx = \frac{i\pi}{e^{i\pi a}-e^{-i\pi a}} = \frac{\pi/2}{(e^{i\pi a}-e^{-i\pi a})/2i}$$

$$\int_0^{\infty} \frac{x^{2a-1}}{x^2+1}\,dx = \frac{\pi/2}{\sin\pi a} = \frac{\pi}{2}\csc\pi a$$

Problem 4.6

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(1+x^2)\,(1-2x\cos\theta+x^2)} + \lim_{\rho\to\infty}\int_0^{\pi} \frac{(\rho e^{i\theta})^2\, ie^{i\theta}\,d\theta}{(1+\rho^2 e^{2i\theta})\,(1-2\rho e^{i\theta}\cos\theta+\rho^2 e^{2i\theta})} = \oint \frac{z^2\,dz}{(z-i)(z+i)(z-e^{i\theta})(z+e^{i\theta})}$$

The line integral along the half-circle at $+\infty$ is zero. The poles within the contour are $z=i$ and $z=e^{i\theta}$.

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(1+x^2)\,(1-2x\cos\theta+x^2)} = 2\pi i\left(f(i)+f(e^{i\theta})\right)$$

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(1+x^2)\,(1-2x\cos\theta+x^2)} = 2\pi i\left\{\frac{i^2}{2i(i-e^{i\theta})(i-e^{-i\theta})} + \frac{e^{2i\theta}}{(e^{i\theta}-i)(e^{i\theta}+i)(e^{i\theta}-e^{-i\theta})}\right\}$$

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(1+x^2)\,(1-2x\cos\theta+x^2)} = 2\pi i\left\{\frac{-1}{2i(-1-ie^{i\theta}-ie^{-i\theta}+1)} + \frac{e^{2i\theta}}{(e^{2i\theta}+1)(e^{i\theta}-e^{-i\theta})}\right\}$$

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(1+x^2)\,(1-2x\cos\theta+x^2)} = 2\pi i\left\{\frac{i}{2(-i)(e^{i\theta}+e^{-i\theta})} + \frac{e^{i\theta}}{(e^{i\theta}+e^{-i\theta})(e^{i\theta}-e^{-i\theta})}\right\}$$

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(1+x^2)\,(1-2x\cos\theta+x^2)} = 2\pi i\left\{\frac{-1}{4\cos\theta} + \frac{e^{i\theta}}{4i\cos\theta\sin\theta}\right\}$$

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(1+x^2)\,(1-2x\cos\theta+x^2)} = \frac{\pi i}{2\cos\theta}\left\{-1+\frac{\cos\theta+i\sin\theta}{i\sin\theta}\right\}$$

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(1+x^2)\,(1-2x\cos\theta+x^2)} = \frac{\pi i}{2\cos\theta}\left\{-i\cot\theta\right\} = \frac{\pi}{2\sin\theta}$$

Problem 4.7

$$\int_{-\infty}^{0} \frac{x^a\,dx}{(1+x^2)^2} = \int_{\infty}^{0} \frac{(-1)^a x^a\,d(-x)}{(1+(-x)^2)^2} = \int_{0}^{\infty} \frac{(-1)^a x^a\,dx}{(1+x^2)^2}$$

$$\int_{0}^{\infty} \frac{(-1)^a x^a\,dx}{(1+x^2)^2} + \int_{0}^{\infty} \frac{x^a\,dx}{(1+x^2)^2} + \lim_{\rho\to\infty} \int_{0}^{\pi} \frac{(\rho e^{i\theta})^a\,ie^{i\theta}\,d\theta}{(1+\rho^2 e^{2i\theta})^2} = \oint \frac{z^a\,dz}{(1+z^2)^2}$$

The line integral along the half-circle at $\rho \to \infty$ is zero. The only pole within the contour is $z = i$.

$$(1+(-1)^a)\int_{0}^{\infty} \frac{x^a\,dx}{(1+x^2)^2} = \oint \frac{z^a\,dz}{(1+z^2)^2} = 2\pi f'(i)$$

$$f = \frac{z^a}{(z+i)2}$$

$$f' = \frac{az^{a-1}}{(z+i)^2} - \frac{2z^a}{(z+i)^3}$$

$$f' = \frac{az^{a-1}(z+i) - 2z^a}{(z+i)^3}$$

$$f'(i) = \frac{ai^{a-1}(2i) - 2i^a}{(2i)^3}$$

$$f'(i) = \frac{2ai^a - 2i^a}{8i^3} = \frac{(a-1)i^a}{4i^3} = \frac{a-1}{4}\,i^{a-3}$$

$$(1+(-1)^a)\int_{0}^{\infty} \frac{x^a\,dx}{(1+x^2)^2} = 2\pi i\left(\frac{a-i}{4}\,i^{a-3}\right) = \frac{(a-1)\pi}{2}\,i^{a-2}$$

$$\int_{0}^{\infty} \frac{x^a\,dx}{(1+x^2)^2} = \frac{(a-1)\pi i^{a-2}}{2(1+e^{i\pi a})} = \frac{(a-1)\pi e^{\frac{i\pi a}{2}}(-1)}{2(1+e^{i\pi a})}$$

$$\int_{0}^{\infty} \frac{x^a\,dx}{(1+x^2)^2} = \frac{(1-a)\pi}{4\left\{\frac{e^{-\frac{i\pi a}{2}}+e^{\frac{i\pi a}{2}}}{2}\right\}} = \frac{(1-a)\pi}{4\cos(\frac{\pi a}{2})}$$

Problem 4.10

$$\int_0^\infty \frac{x^{-a}\,dx}{1+2x\cos\theta+x^2} = \int_0^\infty \frac{x^{-(a-1)-1}\,dx}{1+2x\cos\theta+x^2}$$

$$\int_0^\infty \frac{x^{-a}\,dx}{1+2x\cos\theta+x^2} = \frac{\pi}{\sin\{-(a-1)\pi\}} \times \text{residues of all poles of} \quad \frac{(-z)^{-a}}{a+2z\cos\theta+z^2}$$

$$z^2+2z\cos\theta+1=0$$

$$z = \frac{-2\cos\theta \pm \sqrt{4\cos^2\theta-4}}{2}$$

$$z = -\cos\theta \pm i\sin\theta = -e^{\pm i\theta}$$

$$\int_0^\infty \frac{x^{-a}\,dx}{1+2x\cos\theta+x^2} = \frac{\pi}{\sin\{-(a-1)\pi\}} \left\{ \frac{(e^{i\theta})^{-a}}{-e^{i\theta}+e^{-i\theta}} + \frac{(e^{-i\theta})^{-a}}{-e^{-i\theta}+e^{i\theta}} \right\}$$

$$\int_0^\infty \frac{x^{-a}\,dx}{1+2x\cos\theta+x^2} = \frac{\pi}{-\sin a\pi\cos\pi+\cos a\pi\sin\pi} \left\{ \frac{-e^{-ia\theta}+e^{ia\theta}}{e^{i\theta}-e^{-i\theta}} \right\}$$

$$\int_0^\infty \frac{x^{-a}\,dx}{1+2x\cos\theta+x^2} = \frac{\pi}{\sin a\pi}\,\frac{\sin a\theta}{\sin\theta}$$

Problem 4.12

$$\int_{-\infty}^{\infty} \frac{\cos(px) - \cos(qx)}{x^2} = \lim_{\xi \to 0} \int_{-\infty}^{\infty} \frac{\cos(px) - \cos(qx)}{x^2 - \xi^2}\, dx$$

$$\int_{-\infty}^{\infty} \frac{\cos(px) - \cos(qx)}{x^2} = \lim_{\xi \to 0} \int_{-\infty}^{\infty} \frac{\cos(px)}{x^2 - \xi^2}\, dx - \lim_{\xi \to 0} \int_{-\infty}^{\infty} \frac{\cos(qx)}{x^2 - \xi^2}\, dx$$

From the previous problem,

$$\lim_{\xi \to 0} \int_{-\infty}^{\infty} \frac{\cos(px)}{x^2 - \xi^2}\, dx = -\frac{\pi}{\xi} \sin p\xi$$

$$\int_{-\infty}^{\infty} \frac{\cos(px) - \cos(qx)}{x^2} = \lim_{\xi \to 0} \left\{ -\pi \frac{\sin(p\xi)}{\xi}\, dx + \pi \frac{\sin(q\xi)}{\xi}\, dx \right\}$$

$$\int_{-\infty}^{\infty} \frac{\cos(px) - \cos(qx)}{x^2} = \lim_{\xi \to 0} \left\{ -\pi \frac{p\xi - \frac{p^3\xi^3}{3!} + \dots}{\xi} + \pi \frac{q\xi - \frac{q^3\xi^3}{3!} + \dots}{\xi} \right\}$$

$$\int_{-\infty}^{\infty} \frac{\cos(px) - \cos(qx)}{x^2} = \lim_{\xi \to 0} \left\{ \pi(q - p) + \pi \frac{(p^3 - q^3)\xi^2}{3!} - \dots \right\}$$

$$\int_{-\infty}^{\infty} \frac{\cos(px) - \cos(qx)}{x^2} = \pi(q - p)$$

Problem 4.13

$$f(z) = \frac{e^{iz}}{(z+ia)(z+ib)} = \frac{e^{i(x+iy)}}{\left((x+i(a+y))\left(x+i(b+y)\right)\right)}$$

$$f(z) = \frac{e^{-y}e^{ix}\left(x-i(a+y)\right)\left(x-i(b+y)\right)}{\left(x^2+(a+y)^2\right)\left(x^2+(b+y)^2\right)}$$

$$f(z) = \frac{e^{-y}(\cos x + i\sin y)\left\{\left(x^2-(a+y)(b+y)\right) - ix(a+b+2y))\right\}}{\left(x^2+(a+y)^2\right)\left(x^2+(b+y)^2\right)}$$

$$u(x,y) = \frac{e^{-y}\cos x\left(x^2-(a+y)(b+y)\right) + e^{-y}x\sin x(a+b+2y)}{\left(x^2+(a+y)^2\right)\left(x^2+(b+y)^2\right)}$$

$$u(x,0) = \frac{\cos x\left(x^2-ab)\right) + x\sin x(a+b)}{\left(x^2+a^2\right)\left(x^2+b^2\right)}$$

$$v(x,y) = \frac{e^{-y}\sin x\left(x^2-(a+y)(b+y)\right) - e^{-y}x\cos x(a+b+2y)}{\left(x^2+(a+y)^2\right)\left(x^2+(b+y)^2\right)}$$

$$v(x,0) = \frac{\sin x\left(x^2-ab\right) - x\cos x(a+b)}{\left(x^2+a^2\right)\left(x^2+b^2\right)}$$

$$\int_{-\infty}^{\infty} \frac{-(x^2-ab)\sin x + (a+b)x\cos x}{x(x^2+a^2)(x^2+b^2)}\,dx = -\int_{-\infty}^{\infty} \frac{v(x,0)-v(0,0)}{x-0}\,dx$$

$$\int_{-\infty}^{\infty} \frac{-(x^2-ab)\sin x + (a+b)x\cos x}{x(x^2+a^2)(x^2+b^2)}\,dx = -\pi u(0,0)$$

$$\int_{-\infty}^{\infty} \frac{-(x^2-ab)\sin x + (a+b)x\cos x}{x(x^2+a^2)(x^2+b^2)}\,dx = -\pi\left(-\frac{ab}{a^2b^2}\right)$$

$$\int_{-\infty}^{\infty} \frac{-(x^2-ab)\sin x + (a+b)x\cos x}{x(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{ab}$$

Problem 4.14

The MacLaurin series expansion of $\tan^{-1} x$ is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\frac{1}{x}\tan^{-1} x = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \dots$$

$$1 - \frac{1}{x}\tan^{-1} x = \frac{x^2}{3} - \frac{x^4}{5} + \frac{x^6}{7} + \dots$$

$$\frac{1}{x^2}\left(1 - \frac{1}{x}\tan^{-1} x\right) = \frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} - \dots$$

$$(1+x^2)\frac{1}{x^2}\left(1 - \frac{1}{x}\tan^{-1} x\right) = \frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} - \dots + \frac{x^3}{3} - \frac{x^4}{5} + \dots$$

$$(1+x^2)\frac{1}{x^2}\left(1 - \frac{1}{x}\tan^{-1} x\right) = \frac{1}{3} + \frac{2x^2}{15} - \frac{2x^4}{35} + \dots$$

$$\lim_{x\to 0} \ln\left\{\left(\frac{1+x^2}{x^2}\right)\left(1 - \frac{\tan^{-1} x}{x}\right)\right\} = \ln\left(\frac{1}{3}\right) = -\ln 3$$

The integrand $\frac{1}{z}\ln\left\{\left(\frac{1+z^2}{z^2}\right)\left(1 - \frac{\tan^{-1} z}{z^2}\right)\right\}$ has a first-order pole at $z = 0$. It is otherwise analytic throughout the region $-1 < \mathrm{Im(z)} < 1$.

$$\oint \frac{1}{z}\ln\left\{\left(\frac{1+z^2}{z^2}\right)\left(1 - \frac{\tan^{-1} z}{z}\right)\right\} dz = 2\pi i \lim_{z\to 0} \ln\left\{\left(\frac{1+z^2}{z^2}\right)\left(1 - \frac{\tan^{-1} z}{z}\right)\right\}$$

$$\int_{-\infty-i\beta}^{\infty-i\beta} \frac{1}{z}\ln\left\{\left(\frac{1+z^2}{z^2}\right)\left(1 - \frac{\tan^{-1} z}{z}\right)\right\} dz + \int_{\infty+i\beta}^{-\infty+i\beta} \frac{1}{z}\ln\left\{\left(\frac{1+z^2}{z^2}\right)\left(1 - \frac{\tan^{-1} z}{z}\right)\right\} dz = -2\pi i\ln 3$$

$$\int_{-\infty-i\beta}^{\infty-i\beta} \frac{1}{z}\ln\left\{\left(\frac{1+z^2}{z^2}\right)\left(1 - \frac{\tan^{-1} z}{z}\right)\right\} dz - \int_{-\infty+i\beta}^{\infty+i\beta} \frac{1}{z}\ln\left\{\left(\frac{1+z^2}{z^2}\right)\left(1 - \frac{\tan^{-1} z}{z}\right)\right\} dz = -2\pi i\ln 3$$

$$\int_{-\infty-i\beta}^{\infty-i\beta} \frac{1}{z}\ln\left\{\left(\frac{1+z^2}{z^2}\right)\left(1 - \frac{\tan^{-1} z}{z}\right)\right\} dz = \int_{-(-\infty-i\beta)}^{-(\infty-i\beta)} \frac{1}{(-z')}\ln\left\{\left(\frac{1+(-z')^2}{(-z')^2}\right)\left(1 - \frac{\tan^{-1}(-z')}{(-z')}\right)\right\} d(-z')$$

$$\int_{-\infty-i\beta}^{\infty-i\beta} \frac{1}{z}\ln\left\{\left(\frac{1+z^2}{z^2}\right)\left(1 - \frac{\tan^{-1} z}{z}\right)\right\} dz = \int_{\infty+i\beta}^{-\infty+i\beta} \frac{1}{z'}\ln\left\{\left(\frac{1+z'^2}{z'^2}\right)\left(1 - \frac{\tan^{-1} z'}{z'}\right)\right\} dz'$$

$$\int_{-\infty-i\beta}^{\infty-i\beta} \frac{1}{z}\ln\left\{\left(\frac{1+z^2}{z^2}\right)\left(1 - \frac{\tan^{-1} z}{z}\right)\right\} dz = -\int_{-\infty+i\beta}^{\infty+i\beta} \frac{1}{z'}\ln\left\{\left(\frac{1+z'^2}{z'^2}\right)\left(1 - \frac{\tan^{-1} z'}{z'}\right)\right\} dz'$$

$$\int_{-\infty-i\beta}^{\infty-i\beta} \frac{1}{z}\ln\left\{\left(\frac{1+z^2}{z^2}\right)\left(1 - \frac{\tan^{-1} z}{z}\right)\right\} dz - \int_{\infty+i\beta}^{-\infty+i\beta} \frac{1}{z}\ln\left\{\left(\frac{1+z^2}{z^2}\right)\left(1 - \frac{\tan^{-1} z}{z}\right)\right\} dz = -2\pi\ln 3$$

$$2\int_{-\infty-i\beta}^{\infty-i\beta} \frac{1}{z}\ln\left\{\left(\frac{1+z^2}{z^2}\right)\left(1 - \frac{\tan^{-1} z}{z}\right)\right\} dz = -2\pi\ln 3$$

$$\frac{1}{2\pi i}\int_{-\infty-i\beta}^{\infty-i\beta} \frac{1}{z}\ln\left\{\left(\frac{1+z^2}{z^2}\right)\left(1 - \frac{\tan^{-1} z}{z}\right)\right\} dz = -\frac{1}{2}\ln 3$$

Problem 4.15

$$I(z) = \int_0^\infty \frac{f(t)}{t-z}\,dt = \lim_{\epsilon\to 0}\left\{\int_0^{z-\epsilon} \frac{f(t)}{t-z}\,dt + \int_{-\pi}^{0} \frac{f(z+\epsilon e^{i\theta})}{(z+\epsilon e^{i\theta})-z}\,d(z+\epsilon e^{i\theta}) + \int_{z+\epsilon}^\infty \frac{f(t)}{t-z}\,dt\right\}$$

$$I(z) = \lim_{\epsilon\to 0}\int_{-\pi}^{0} \frac{f(z+\epsilon e^{i\theta})}{\epsilon e^{i\theta}}\,i\epsilon e^{i\theta}\,d\theta + \lim_{\epsilon\to 0}\left\{\int_0^{z-\epsilon} \frac{f(t)}{t-z}\,dt + \int_{z+\epsilon}^\infty \frac{f(t)}{t-z}\,dt\right\}$$

$$I(z) = \lim_{\epsilon\to 0}\int_{-\pi}^{0} \left\{if(z)\,d\theta + if'(z)\epsilon e^{i\theta}\,d\theta + \ldots\right\} + \mathcal{P}\int_0^\infty \frac{f(t)}{t-z}\,dt$$

$$I(z) = if(z)\int_{-\pi}^{0} d\theta + \lim_{\epsilon\to 0} i\epsilon\int_{-\pi}^{0} f'(z)e^{i\theta}\,d\theta + \ldots + \mathcal{P}\int_0^\infty \frac{f(t)}{t-z}\,dt$$

$$I(z) = i\pi f(z) + \mathcal{P}\int_0^\infty \frac{f(t)}{t-z}\,dt$$

Problem 4.18

$$f(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2})(-z)^n}{\Gamma(1+n)\Gamma(\frac{1}{2}-n)}$$

The radius of convergence is

$$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\Gamma(\frac{1}{2})(-1)^n}{\Gamma(1+n)\Gamma(\frac{1}{2}-n)} \Bigg/ \frac{\Gamma(\frac{1}{2})(-1)^{n+1}}{\Gamma(2+n)\Gamma(-\frac{1}{2}-n)}$$

$$R = \lim_{n\to\infty} (-1)\frac{\Gamma(2+n)}{\Gamma(1+n)} \frac{\Gamma(-\frac{1}{2}-n)}{\Gamma(\frac{1}{2}-n)}$$

$$R = \lim_{n\to\infty} = (-1)\frac{(1+n)\Gamma(1+n)}{\Gamma(1+n)} \frac{\Gamma(-\frac{1}{2}-n)}{(\frac{1}{2}-n)\Gamma(-\frac{1}{2}-n)}$$

$$R = \lim_{n\to\infty} -\frac{1+n}{\frac{1}{2}-n} = 1$$

For n odd

$$\Gamma(\tfrac{1}{2}) > 0 \qquad \Gamma(1+n) > 0 \qquad \Gamma(\tfrac{1}{2}-n) < 0$$

$$\Gamma(\tfrac{1}{2}-1) = \Gamma(-\tfrac{1}{2}) = \tfrac{\Gamma(\frac{1}{2})}{-\frac{1}{2}} < 0$$

$$\Gamma(\tfrac{1}{2}-2) = \Gamma(-\tfrac{3}{2}) = \tfrac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}} > 0$$

$$\Gamma(\tfrac{1}{2}-n-1) = \Gamma(-\tfrac{1}{2}-n) = \tfrac{\Gamma(\frac{1}{2})}{-\frac{1}{2}-n-1}$$

$$\Gamma(\tfrac{1}{2}-n-1) = -(n+1-\tfrac{1}{2})\Gamma(-\tfrac{1}{2}-n)$$

Since $n+1 > \frac{1}{2}$, $\Gamma(\frac{1}{2}-n)$ changes sign for each increment in n.

$$n=0 \to \Gamma(\frac{1}{2}-n) > 0 \quad n=1 \to \Gamma(\frac{1}{2}-n) < 0 \quad n=2m \to \Gamma(\frac{1}{2}-2m) > 0 \quad n=1+2m \to \Gamma(\frac{1}{2}-1-2m) < 0$$

$$b_n = \frac{\Gamma(\frac{1}{2})(-1)^n}{\Gamma(1+n)\Gamma(\frac{1}{2}-n)} = \frac{\Gamma(\frac{1}{2})(-1)^n}{\Gamma(1+n)(-1)^n|\Gamma(\frac{1}{2}-n)|} = \frac{\Gamma(\frac{1}{2})}{\Gamma(1+n)|\Gamma(\frac{1}{2}-n)|} > 0$$

for all n. From the theorem at the top of page 387, $z=1$ is a singular point.

Problem 4.19

$$b_n = \prod_i \frac{(a_i+n-1)!}{(a_i-1)!} \prod_j \frac{(c_j-1)!}{(c_j+n-1)!} \frac{1}{n!}$$

$$R = \lim_{n\to\infty} \frac{b_n}{b_{n+1}} = \left\{ \frac{n!}{(n+1)!} \right\}^{-1} \prod_i \frac{(a_i+n-1)!}{(a_i+n)!} \prod_j \frac{(c_j+n)!}{(c_j+n-1)!}$$

$$R = \lim_{n\to\infty} n \prod_i \frac{1}{(a_i+n)} \prod_j (c_j+n)$$

$$R \to \infty$$

Assuming $a_j > 0$ and $c_j > 0$, then Re $b_n > 0$ for all n and by the theorem at the top of page 387, F is singular at $z = 1$.

$$b_n = \prod_{i=1}^{s} \frac{(a_0+n-1)!}{(a_0-1)!} \frac{(a_i+n-1)!}{(a_i-1)!} \frac{(c_i-1)!}{(c_i+n-1)!} \frac{1}{n!}$$

$$b_n = \prod_{i=1}^{s} \frac{(c_i-1)!}{(a_0-1)!(a_i-1)!} \frac{(a_0+n-1)!}{n!} \frac{(a_i+n-1)!}{(c_i+n-1)!}$$

$$b_n = \prod_{i=1}^{s} \frac{(c_i-1)!}{(a_0-1)!(a_i-1)!} \frac{(n+1)(n+2)\ldots(a_0+n-1)}{(a_i+n)(a_i+n+1)\ldots(c_i+n-1)}$$

$$\lim_{n\to\infty} b_n = \prod_{i=1}^{s} \frac{(c_i-1)!}{(a_0-1)!(a_i-1)!} \frac{n^{a_0-1}}{n^{c_i-a_i}} = \left\{ \prod_{i=1}^{s} \frac{(c_i-1)!}{(a_0-1)!(a_i-1)!} \right\} n^{a_0-1+\sum_{i=1}^{s} a_i-c_i}$$

$$\lim_{n\to\infty} b_n = \left\{ \prod_{i=1}^{s} \frac{\Gamma(c_i)}{\Gamma(a_0)\Gamma(a_i)} \right\} n^{a_0-1+\sum_{i=1}^{s} a_i-c_i}$$

Let $p = a_0 + \sum_{i=1}^{s} a_i - c_i$; $b_n = \left\{ \prod_{i=1}^{s} \frac{\Gamma(c_i)}{\Gamma(a_0)\Gamma(a_i)} \right\} \Gamma(p) \frac{n^{p-1}}{(p-1)!}$.

$$F(a_0, a_i|c_i|z) \xrightarrow[z\to 1]{} \left\{ \prod_{i=1}^{s} \frac{\Gamma(c_i)}{\Gamma(a_0)\Gamma(a_i)} \right\} \Gamma(p) \frac{n^{p-1}}{(p-1)!}$$

per remarks at top of page 388.

Problem 4.21

From the solution to Problem 4.24 below:

$$\frac{\pi \coth(\pi a)}{a} = \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2}$$

$$\frac{\pi \coth(\pi b)}{b} = \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + b^2}$$

$$\pi \left(\frac{\coth(\pi a)}{a} - \frac{\coth(\pi b)}{b} \right) = \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} - \frac{1}{n^2 + b^2}$$

$$\pi \left(\frac{\coth(\pi a)}{a} - \frac{\coth(\pi b)}{b} \right) = \sum_{n=-\infty}^{\infty} \frac{(n^2 + b^2) - (n^2 + a^2)}{(n^2 + b^2)(n^2 + a^2)}$$

$$\pi \left(\frac{\coth(\pi a)}{a} - \frac{\coth(\pi b)}{b} \right) = \sum_{n=-\infty}^{\infty} \frac{(b^2 - a^2)}{(n^2 + b^2)(n^2 + a^2)}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n^2 + b^2)(n^2 + a^2)} = \frac{\pi}{b^2 - a^2} \left(\frac{\coth(\pi a)}{a} - \frac{\coth(\pi b)}{b} \right)$$

Problem 4.23

$$e^{az}-e^{bz}=e^{\frac{az}{2}}e^{\frac{bz}{2}}\left(e^{\frac{az}{2}-\frac{bz}{2}}-e^{\frac{bz}{2}-\frac{az}{2}}\right)$$

$$e^{az}-e^{bz}=e^{\frac{1}{2}(a+b)z}\left(e^{\frac{1}{2}(a-b)z}-e^{\frac{1}{2}(a-b)z}\right)$$

$$e^{az}-e^{bz}=e^{\frac{1}{2}(a+b)z}2\sinh\frac{1}{2}(a-b)z$$

$$e^{az}-e^{bz}=e^{\frac{1}{2}(a+b)z}2\left\{\frac{1}{2}(a-b)z\right\}\frac{\sinh\frac{1}{2}(a-b)z}{\frac{1}{2}(a-b)z}$$

$$e^{az}-e^{bz}=e^{\frac{1}{2}(a+b)z}(a-b)z\frac{\sinh\frac{1}{2}(a-b)z}{\frac{1}{2}(a-b)z}$$

$\dfrac{\sinh w}{w}$ is an even function.

$$\frac{\sinh w}{w}=\frac{e^{w}-e^{-w}}{2w}=\frac{e^{i\left(\frac{w}{i}\right)}-e^{-i\left(\frac{w}{i}\right)}}{2i\left(\frac{w}{i}\right)}=\frac{\sin\left(\frac{w}{i}\right)}{\frac{w}{i}}$$

$$\frac{\sinh w}{w}=\prod_{n=1}^{\infty}\left(1-\frac{\left(\frac{w}{i}\right)^2}{n^2\pi^2}\right)=\prod_{n=1}^{\infty}\left(1+\frac{w^2}{n^2\pi^2}\right)$$

$$\frac{\sinh\frac{1}{2}(a-b)z}{\frac{1}{2}(a-b)z}=\prod_{n=1}^{\infty}\left(1+\frac{\frac{1}{4}(a-b)^2z^2}{n^2\pi^2}\right)$$

$$e^{az}-e^{bz}=e^{\frac{1}{2}(a+b)z}(a-b)z\prod_{n=1}^{\infty}\left(1+\frac{(a-b)^2z^2}{4n^2\pi^2}\right)$$

Problem 4.30

$$\int_C e^{-t^2} t^{n-\lambda-1}\,dt = \int_C e^{-u}\left(u^{1/2}\right)^{n-\lambda-1}\,\frac{du}{2u^{1/2}} = \frac{1}{2}\int_C e^{-u} u^{\frac{n-\lambda}{2}-1}\,du$$

$$\int_C e^{-t^2} t^{n-\lambda-1}\,dt = \frac{1}{2}(-1)^{\frac{n-\lambda}{2}-1}\int_C e^{-u}(-u)^{\frac{n-\lambda}{2}-1}du$$

$$\int_C e^{-t^2} t^{n-\lambda-1}\,dt = \frac{1}{2}(-1)^{\frac{n-\lambda}{2}-1}(-2\pi i\sin\pi z)\Gamma(z)$$

$$\int_C e^{-t^2} t^{n-\lambda-1}\,dt = (-1)^{\frac{n-\lambda}{2}}\pi i\sin\pi z\Gamma(z)$$

$$\int_C e^{-t^2} t^{n-\lambda-1}\,dt = \pi i^{n+1-\lambda}\sin\pi z\Gamma(z)$$

Problem 4.31

$$1-\frac{\cos\pi\alpha}{\cosh\pi z}=\frac{\cosh\pi z-\cos\pi\alpha}{\cosh\pi z}$$

$$1-\frac{\cos\pi\alpha}{\cosh\pi z}=\frac{\cos i\pi z-\cos\pi\alpha}{\cos i\pi z}$$

$$1-\frac{\cos\pi\alpha}{\cosh\pi z}=\frac{2\left\{\cos^2\frac{i\pi z}{2}-\frac{1}{2}\right\}-2\left\{\cos^2\frac{\pi\alpha}{2}-\frac{1}{2}\right\}}{\sin\left(\frac{\pi}{2}+i\pi z\right)}$$

$$1-\frac{\cos\pi\alpha}{\cosh\pi z}=\frac{2\left\{\cos^2\frac{i\pi z}{2}-\cos^2\frac{\pi\alpha}{2}\right\}}{\sin\left(\frac{\pi}{2}+i\pi z\right)}$$

$$1-\frac{\cos\pi\alpha}{\cosh\pi z}=\frac{2\left\{\cos^2\frac{i\pi z}{2}\sin^2\frac{\pi\alpha}{2}+\cos^2\frac{i\pi z}{2}\cos^2\frac{\pi\alpha}{2}-\sin^2\frac{i\pi z}{2}\cos^2\frac{\pi\alpha}{2}-\cos^2\frac{i\pi z}{2}\cos^2\frac{\pi\alpha}{2}\right\}}{\sin\left(\frac{\pi}{2}+i\pi z\right)}$$

$$1-\frac{\cos\pi\alpha}{\cosh\pi z}=\frac{2\left\{\cos^2\frac{i\pi z}{2}\sin^2\frac{\pi\alpha}{2}-\sin^2\frac{i\pi z}{2}\cos^2\frac{\pi\alpha}{2}\right\}}{\sin\left(\frac{\pi}{2}+i\pi z\right)}$$

$$1-\frac{\cos\pi\alpha}{\cosh\pi z}=\frac{2\left\{\cos\frac{i\pi z}{2}\sin\frac{\pi\alpha}{2}-\sin\frac{i\pi z}{2}\cos\frac{\pi\alpha}{2}\right\}\left\{\cos\frac{i\pi z}{2}\sin\frac{\pi\alpha}{2}+\sin\frac{i\pi z}{2}\cos\frac{\pi\alpha}{2}\right\}}{\sin\left(\frac{\pi}{2}+i\pi z\right)}$$

$$1-\frac{\cos\pi\alpha}{\cosh\pi z}=2\frac{\sin\pi\left(\frac{\alpha-iz)}{2}\right)\sin\pi\left(\frac{\alpha+iz}{2}\right)}{\sin\pi\left(\frac{1}{2}+iz\right)}$$

$$1-\frac{\cos\pi\alpha}{\cosh\pi z}=2\frac{\Gamma\left(\frac{1}{2}+iz\right)\Gamma\left(1-\left(\frac{1}{2}+iz\right)\right)}{\pi}\frac{\pi}{\Gamma\left(\frac{\alpha-iz}{2}\right)\Gamma\left(1-\frac{\alpha-iz}{2}\right)}\frac{\pi}{\Gamma\left(\frac{\alpha+iz}{2}\right)\Gamma\left(1-\frac{\alpha+iz}{2}\right)}$$

$$1-\frac{\cos\pi\alpha}{\cosh\pi z}=\frac{2\pi\Gamma\left(\frac{1}{2}+iz\right)\Gamma\left(\frac{1}{2}-iz\right)}{\Gamma\left(\frac{\alpha+iz}{2}\right)\Gamma\left(\frac{\alpha-iz}{2}\right)\Gamma\left(1-\frac{\alpha+iz}{2}\right)\Gamma\left(1-\frac{\alpha-iz}{2}\right)}$$

Problem 4.32

$$\Gamma(iy)\Gamma(-iy) = \left\{iye^{\gamma(iy)} \prod_{m=1}^{\infty} \left(1+\frac{iy}{m}\right) e^{\frac{-iy}{m}}\right\}^{-1} + \left\{-iye^{\gamma(-iy)} \prod_{m=1}^{\infty} \left(1-\frac{iy}{m}\right) e^{\frac{iy}{m}}\right\}^{-1}$$

$$\Gamma(iy)\left\{\Gamma(iy)\right\}^{*} = \left\{y^2 \prod_{m=1}^{\infty} \left(1+\frac{y^2}{m^2}\right)\right\}^{-1}$$

$$\left|\Gamma(iy)\right|^2 = \left\{y^2 \prod_{m=1}^{\infty} \left(1-\frac{(i\pi y)^2}{m^2\pi^2}\right)\right\}^{-1}$$

$$\left|\Gamma(iy)\right|^2 = \left\{\frac{y}{i\pi}(i\pi y) \prod_{m=1}^{\infty} \left(1-\frac{(i\pi y)^2}{m^2\pi^2}\right)\right\}^{-1}$$

$$\left|\Gamma(iy)\right|^2 = \left\{\frac{y}{i\pi}\sin(i\pi y)\right\}^{-1}$$

$$\left|\Gamma(iy)\right|^2 = \frac{i\pi}{y\sin(i\pi y)} = \frac{\pi}{-iy\sin(i\pi y)} = \frac{\pi}{y\sinh(\pi y)}$$

Problem 4.44

$$\frac{1}{2}\cos^{-1} z^2 = -\frac{i}{2}\ln\left\{z^2 \pm \sqrt{z^4-1}\right\}$$

$$w = \tan\left(\frac{1}{2}\cos^{-1} z^2\right) = -i\frac{\left\{z^2 \pm \sqrt{z^4-1}\right\}^{\frac{1}{2}} - \left\{z^2 \pm \sqrt{z^4-1}\right\}^{-\frac{1}{2}}}{\left\{z^2 \pm \sqrt{z^4-1}\right\}^{\frac{1}{2}} + \left\{z^2 \pm \sqrt{z^4-1}\right\}^{-\frac{1}{2}}}$$

$$w = -i\frac{z^2 \pm \sqrt{z^4-1} - 1}{z^2 \pm \sqrt{z^4-1} + 1} = -i\frac{(z^2-1) \pm \sqrt{(z^2-1)(z^2+1)}}{(z^2+1) \pm \sqrt{(z^2-1)(z^2+1)}}$$

$$w = -i\sqrt{\frac{z^2-1}{z^2+1}}\frac{\sqrt{z^2-1} \pm \sqrt{z^2+1}}{\sqrt{z^2+1} \pm \sqrt{z^2-1}} = \mp i\sqrt{\frac{z^2-1}{z^2+1}} = \mp i\frac{\sqrt{(z-1)(z+1)}}{\sqrt{(z-i)(z+i)}}$$

$$w^2 = -\frac{z^2-1}{z^2+1}$$

$$w^2(z^2+1) = -z^2+1$$

$$z^2(w^2+1) = 1-w^2$$

$$z^2 = \frac{1-w^2}{1+w^2} = \frac{(1-w)^{\frac{1}{2}}(1+w)^{\frac{1}{2}}}{(w+i)^{\frac{1}{2}}(w-i)^{\frac{1}{2}}} = i(w-1)^{\frac{1}{2}}(w+1)^{\frac{1}{2}}(w+i)^{-\frac{1}{2}}(w-i)^{-\frac{1}{2}}$$

The point $z=0$ is at $w=\pm 1$. The points $z=\pm i$ are at infinity. The points $z=\pm 1$ are at $w=0$. Suitable problems: heat or electromagnetic radiation from crossed sources.

Problem 4.48

$$F_-(s) = \int_0^1 f(x)x^{s-1}\,dx$$

Assume that $f(x)$ is analytic for $0 \le x \le 1$. Expand $f(x)$ in a MacLaurin series,

$$f(x) = f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) + \dots$$

$f(x) \to cx^\alpha$ as $x \to 0$ and $f(x)x^{s-1} \to cx^{\alpha+s-1}$. For the integral to converge at the point s,

$$cx^{\alpha+s-1} = cx^{\alpha+\mathrm{Re}(s)-1}x^{i\mathrm{Im}(s)}$$

has to be non-singular for $x \to 0$, so that

$$\alpha + \mathrm{Re}(s) - 1 > 0$$

$$\mathrm{Re}(s) > \sigma_0 = 1 - \alpha$$

$$F_+(s) = \int_1^\infty f(x)x^{s-1}\,dx$$

For $F_+(s)$ to be analytic the integral must converge at the point s. Assuming $f(x)$ is analytic,

$$f(x)x^{s-1} = f(x)x^{\mathrm{Re}(s)-1}x^{i\mathrm{Im}(s)} \to 0 \qquad \text{as} \qquad x \to \infty.$$

Convergence is possible for any $f(x)$ for which

$$f(x) \to cx^\alpha \qquad \text{as} \qquad x \to \infty.$$

$$f(x)x^{s-1} = f(x)x^{\mathrm{Re}(s)-1}x^{i\mathrm{Im}(s)} \to 0 \qquad \text{as} \qquad x \to \infty.$$

$$f(x)x^{s-1} \to cx^{\alpha+\mathrm{Re}(s)-1}x^{i\mathrm{Im}(s)} \to 0$$

if

$$\alpha + \mathrm{Re}(s) - 1 < 0$$

$$\mathrm{Re}(s) < 1 - \alpha - \sigma_1$$

Problem 5.2

$$x = a\xi_3\frac{\sqrt{1-\xi_2^2}}{\xi_1-\xi_2}; \quad y = a\frac{\sqrt{(1-\xi_2^2)(1-\xi_3^2)}}{\xi_1-\xi_2}; \quad z = a\frac{\sqrt{\xi_1^2-1}}{\xi_1-\xi_2}$$

$$\frac{\partial x}{\partial \xi_1} = -a\xi_3\frac{\sqrt{1-\xi_2^2}}{(\xi_1-\xi_2)^2}; \quad \frac{\partial x}{\partial \xi_2} = a\xi_3\frac{(1-\xi_1\xi_2)}{(\xi_1-\xi_2)^2\sqrt{1-\xi_2^2}}; \quad \frac{\partial x}{\partial \xi_3} = a\frac{\sqrt{1-\xi_2^2}}{\xi_1-\xi_2}$$

$$\frac{\partial y}{\partial \xi_1} = -a\frac{\sqrt{(1-\xi_2^2)(1-\xi_3^2)}}{(\xi_1-\xi_2)^2}; \quad \frac{\partial y}{\partial \xi_2} = a\frac{(1-\xi_1\xi_2)\sqrt{1-\xi_3^2}}{(\xi_1-\xi_2)^2\sqrt{1-\xi_2^2}}; \quad \frac{\partial y}{\partial \xi_3} = -a\xi_3\frac{\sqrt{1-\xi_2^2}}{(\xi_1-\xi_2)\sqrt{1-\xi_3^2}}$$

$$\frac{\partial z}{\partial \xi_1} = a\frac{1-\xi_1\xi_2}{(\xi_1-\xi_2)^2\sqrt{\xi_1^2-1}}; \quad \frac{\partial z}{\partial \xi_2} = a\frac{\sqrt{(\xi_1^2-1)}}{(\xi_1-\xi_2)^2}; \quad \frac{\partial z}{\partial \xi_3} = 0$$

$$g_{\xi_1\xi_1} = \left(\frac{\partial x}{\partial \xi_1}\right)^2 + \left(\frac{\partial y}{\partial \xi_1}\right)^2 + \left(\frac{\partial y}{\partial \xi_1}\right)^2$$

$$g_{\xi_1\xi_1} = a^2\xi_3^2\frac{1-\xi_2^2}{(\xi_1-\xi_2)^4} + a^2\frac{(1-\xi_2^2)(1-\xi_3^2)}{(\xi_1-\xi_2)^4} + a^2\frac{(1-\xi_1\xi_2)^2}{(\xi_1-\xi_2)^4(\xi_1^2-1)} = \frac{a^2}{(\xi_1-\xi_2)^2(\xi_1^2-1)}$$

$$g_{\xi_1\xi_2} = \frac{\partial x}{\partial \xi_1}\frac{\partial x}{\partial \xi_2} + \frac{\partial y}{\partial \xi_1}\frac{\partial y}{\partial \xi_2} + \frac{\partial z}{\partial \xi_1}\frac{\partial z}{\partial \xi_2}$$

$$g_{\xi_1\xi_2} = \frac{-a^2\xi_3^2(1-\xi_1\xi_2)}{(\xi_1-\xi_2)^4} + \frac{-a^2(1-\xi_1\xi_2)(1-\xi_3^2)}{(\xi_1-\xi_2)^4} + \frac{a^2(1-\xi_1\xi_2)}{(\xi_1-\xi_2)^4} = 0$$

$$g_{\xi_1\xi_3} = \frac{\partial x}{\partial \xi_1}\frac{\partial x}{\partial \xi_3} + \frac{\partial y}{\partial \xi_1}\frac{\partial y}{\partial \xi_3} + \frac{\partial z}{\partial \xi_1}\frac{\partial z}{\partial \xi_3}$$

$$g_{\xi_1\xi_3} = \frac{-a^2\xi_3(1-\xi_2^2)}{(\xi_1-\xi_2)^3} + \frac{a^2\xi_3(1-\xi_2^2)}{(\xi_1-\xi_2)^3} + 0 = 0$$

$$g_{\xi_2\xi_2} = \left(\frac{\partial x}{\partial \xi_2}\right)^2 + \left(\frac{\partial y}{\partial \xi_2}\right)^2 + \left(\frac{\partial y}{\partial \xi_2}\right)^2$$

$$g_{\xi_2\xi_2} = a^2\xi_3^2\frac{(1-\xi_1\xi_2)^2}{(\xi_1-\xi_2)^4(1-\xi_2^2)} + a^2\frac{(1-\xi_1\xi_2)^2(1-\xi_3^2)}{(\xi_1-\xi_2)^4(1-\xi_2^2)} + a^2\frac{(\xi_1^2-1)}{(\xi_1-\xi_2)^4} = \frac{a^2}{(\xi_1-\xi_2)^2(1-\xi_2^2)}$$

$$g_{\xi_2\xi_3} = \frac{\partial x}{\partial \xi_2}\frac{\partial x}{\partial \xi_3} + \frac{\partial y}{\partial \xi_2}\frac{\partial y}{\partial \xi_3} + \frac{\partial z}{\partial \xi_2}\frac{\partial z}{\partial \xi_3}$$

$$g_{\xi_2\xi_3} = \frac{a\xi_3(1-\xi_1\xi_2)}{(\xi_1-\xi_2)^3} + \frac{-a\xi_3(1-\xi_1\xi_2)}{(\xi_1-\xi_2)^3} + 0 = 0$$

$$g_{\xi_3\xi_3} = \left(\frac{\partial x}{\partial \xi_3}\right)^2 + \left(\frac{\partial y}{\partial \xi_3}\right)^2 + \left(\frac{\partial y}{\partial \xi_3}\right)^2$$

$$g_{\xi_3\xi_3} = a^2\frac{1-\xi_2^2}{(\xi_1-\xi_2)^2} + a^2\xi_3^2\frac{(1-\xi_2^2)}{(\xi_1-\xi_2)^2(1-\xi_3^2)} + 0 = \frac{a^2(1-\xi_2^2)}{(\xi_1-\xi_2)^2(1-\xi_3^2)}$$

$$g^{\xi_1\xi_1} = \frac{(\xi_1-\xi_2)^2(\xi_1^2-1)}{a^2}; \quad g^{\xi_2\xi_2} = \frac{(\xi_1-\xi_2)^2(1-\xi_2^2)}{a^2}; \quad g^{\xi_3\xi_3} = \frac{(\xi_1-\xi_2)^2(1-\xi_3^2)}{a^2(1-\xi_2^2)}$$

$$g = g_{\xi_1\xi_1}g_{\xi_2\xi_2}g_{\xi_3\xi_3} = \frac{a^6}{(\xi_1-\xi_2)^6(\xi_1^2-1)(1-\xi_3^2)}$$

$$\nabla^2\psi = \left(g^{ij}\psi_{,i}\right)_{,j} + g^{ij}\frac{g_{,i}}{2g}\psi_{,j} = 0$$

$$0 = \left(g^{\xi_1\xi_1}\psi_{,\xi_1}\right)_{,\xi_1} + \left(g^{\xi_2\xi_2}\psi_{,\xi_2}\right)_{,\xi_2} + \left(g^{\xi_3\xi_3}\psi_{,\xi_3}\right)_{,\xi_3} + g^{\xi_1\xi_1}\frac{g_{\xi_1}}{2g}\psi_{,\xi_1} + g^{\xi_1\xi_2}\frac{g_{\xi_2}}{2g}\psi_{,\xi_2} + g^{\xi_1\xi_3}\frac{g_{\xi_3}}{2g}\psi_{,\xi_3}$$

$$0 = g^{\xi_1\xi_1}\psi_{,\xi_1\xi_1} + g^{\xi_2\xi_2}\psi_{,\xi_2\xi_2} + g^{\xi_3\xi_3}\psi_{,\xi_3\xi_3}$$

$$+\psi_{,\xi_1}\left(g^{\xi_1\xi_1}{}_{,\xi_1} + g^{\xi_1\xi_1}\frac{g_{,\xi_1}}{2g}\right) + \psi_{,\xi_2}\left(g^{\xi_2\xi_2}{}_{,\xi_2} + g^{\xi_2\xi_2}\frac{g_{,\xi_2}}{2g}\right) + \psi_{,\xi_3}\left(g^{\xi_3\xi_3}{}_{,\xi_3} + g^{\xi_3\xi_3}\frac{g_{,\xi_3}}{2g}\right)$$

$$0 = \frac{(\xi_1-\xi_2)^2}{a^2}\left\{(\xi_1^2-1)\psi_{,\xi_1\xi_1} + (1-\xi_2^2)\psi_{,\xi_2\xi_2} + \frac{(1-\xi_3^2)}{(1-\xi_2^2)}\psi_{,\xi_3\xi_3}\right\} +$$

$$\psi_{,\xi_1}\left\{\frac{2(\xi_1-\xi_2)(\xi_1^2-1)+2\xi_1(\xi_1-\xi_2)^2}{a^2} - \frac{(\xi_1-\xi_2)^2(\xi_1^2-1)}{a^2}\left(\frac{3}{(\xi_1-\xi_2)} + \frac{\xi_1}{\xi_1^2-1}\right)\right\} +$$

$$\psi_{,\xi_2}\left\{\frac{-2(\xi_1-\xi_2)(1-\xi_2^2)-2\xi_2(\xi_1-\xi_2)^2}{a^2} + \frac{(\xi_1-\xi_2)^2(1-\xi_2^2)}{a^2}\frac{3}{(\xi_1-\xi_2)}\right\} +$$

$$\psi_{,\xi_3}\left\{\frac{-2\xi_3(\xi_1-\xi_2)^2}{a^2(1-\xi_2^2)} + \frac{(\xi_1-\xi_2)^2(1-\xi_3^2)}{a^2(1-\xi_2^2)}\frac{\xi_3}{1-\xi_3^2}\right\}$$

$$0 = (\xi_1^2-1)\psi_{,\xi_1\xi_1} + (1-\xi_2^2)\psi_{,\xi_2\xi_2} + \frac{(1-\xi_3^2)}{(1-\xi_2^2)}\psi_{,\xi_3\xi_3} + \frac{(1-\xi_1\xi_2)}{(\xi_1-\xi_2)}\psi_{,\xi_1} + \frac{(1-2\xi_1\xi_2+\xi_2^2)}{(\xi_1-\xi_2)}\psi_{,\xi_2} - \frac{\xi_3}{1-\xi_2^2}\psi_{,\xi_3}$$

$$0 = (\xi_1^2-1)(1-\xi_2^2)\psi_{,\xi_1\xi_1} + \frac{(1-\xi_1\xi_2)(1-\xi_2^2)}{(\xi_1-\xi_2)}\psi_{,\xi_1} + (1-\xi_2^2)^2\psi_{,\xi_2\xi_2} + \frac{(1-2\xi_1\xi_2+\xi_2^2)(1-\xi_2^2)}{(\xi_1-\xi_2)}\psi_{,\xi_2} + (1-\xi_3^2)\psi_{,\xi_3\xi_3} - \xi_3\psi_{,\xi_3}$$

Let $\psi = X(\xi_1,\xi_2)X_3(\xi_3)$.

$$0 = (\xi_1^2-1)(1-\xi_2^2)X_{,\xi_1\xi_1}X_3 + \frac{(1-\xi_1\xi_2)(1-\xi_2^2)}{(\xi_1-\xi_2)}X_{,\xi_1}X_3 + (1-\xi_2^2)^2X_{,\xi_2\xi_2}X_3$$

$$+\frac{(1-2\xi_1\xi_2+\xi_2^2)(1-\xi_2^2)}{(\xi_1-\xi_2)}X_{,\xi_2}X_3 + (1-\xi_3^2)XX_{3,\xi_3\xi_3} - \xi_3XX_{3,\xi_3}$$

$$0 = (1-\xi_2^2)(\xi_1^2-1)\frac{X_{,\xi_1\xi_1}}{X} + + \frac{(1-\xi_1\xi_2)}{(\xi_1-\xi_2)(\xi_1^2-1)}\frac{X_{,\xi_1}}{X} + (1-\xi_2^2)^2\frac{X_{,\xi_2\xi_2}}{X} + \frac{(1-2\xi_1\xi_2+\xi_2^2)}{(\xi_1-\xi_2)(1-\xi_2^2)}\frac{X_{,\xi_2}}{X}$$

$$+(1-\xi_3^2)\frac{X_{\xi_3\xi_3}}{X} - \xi_3\frac{X,_{\xi_3}}{X}$$

$$(1-\xi_3^2)\frac{X_{\xi_3\xi_3}}{X} - \xi_3\frac{X,_{\xi_3}}{X} = k_3^2$$

Let $X = YR$.

$$X,_{\xi_1} = Y,_{\xi_1} R + YR,_{\xi_1} \qquad X,_{\xi_1\xi_1} = Y,_{\xi_1\xi_1} + 2Y,_{\xi_1} R,_{\xi_1} + YR,_{\xi_1\xi_1}$$

$$X,_{\xi_2} = Y,_{\xi_2} R + YR,_{\xi_2} \qquad X,_{\xi_2\xi_2} = X,_{\xi_2\xi_2} + 2Y,_{\xi_2} R,_{\xi_2} + YR,_{\xi_2\xi_2}$$

$$0 = (1-\xi_2^2)(\xi_1^2-1)\left\{\frac{Y,_{\xi_1\xi_1}}{Y} + \left(2\frac{R,_{\xi_1}}{R} + \frac{(1-\xi_1\xi_2)}{(\xi_1-\xi_2)(\xi_1^2-1)}\right)\frac{Y,_{\xi_1}}{Y}\right\} + (1-\xi_2^2)^2\left\{\frac{Y,_{\xi_2\xi_2}}{Y} + \left(2\frac{R,_{\xi_2}}{R} + \frac{(1-2\xi_1\xi_2+\xi_2^2)}{(\xi_1-\xi_2)(1-\xi_2^2)}\right)\frac{Y,_{\xi_2}}{Y}\right\}$$

$$+\left\{(1-\xi_2^2)(\xi_1^2-1)\frac{R,_{\xi_1\xi_1}}{R} + (1-\xi_2^2)^2\frac{R,_{\xi_2\xi_2}}{R} + \frac{(1-\xi_1\xi_2)(1-\xi_2^2)}{(\xi_1-\xi_2)}\frac{R,_{\xi_1}}{R} + \frac{(1-2\xi_1\xi_2+\xi_2^2)(1-\xi_2^2)}{(\xi_1-\xi_2)}\frac{R,_{\xi_2}}{R}\right\} + k_3^2$$

$$0 = (\xi_1^2-1)\left\{\frac{Y,_{\xi_1\xi_1}}{Y} + \left(2\frac{R,_{\xi_1}}{R} + \frac{(1-\xi_1\xi_2)}{(\xi_1-\xi_2)(\xi_1^2-1)}\right)\frac{Y,_{\xi_1}}{Y}\right\} + (1-\xi_2^2)\left\{\frac{Y,_{\xi_2\xi_2}}{Y} + \left(2\frac{R,_{\xi_2}}{R} + \frac{(1-2\xi_1\xi_2+\xi_2^2)}{(\xi_1-\xi_2)(1-\xi_2^2)}\right)\frac{Y,_{\xi_2}}{Y}\right\}$$

$$+(\xi_1^2-1)\frac{R,_{\xi_1\xi_1}}{R} + (1-\xi_2^2)\frac{R,_{\xi_2\xi_2}}{R} + \frac{(1-\xi_1\xi_2)}{(\xi_1-\xi_2)}\frac{R,_{\xi_1}}{R} + \frac{(1-2\xi_1\xi_2+\xi_2^2)}{(\xi_1-\xi_2)}\frac{R,_{\xi_2}}{R} + \frac{k_3^2}{(1-\xi_2^2)}$$

For the equation in Y to be separable,

$$u(\xi_1) = 2\frac{R,_{\xi_1}}{R} + \frac{(1-\xi_1\xi_2)}{(\xi_1-\xi_2)(\xi_1^2-1)} = \left\{\ln\frac{R^2(\xi_1^2-1)^{\frac{1}{2}}}{\xi_1-\xi_2}\right\},_{\xi_1}$$

$$R^2 = \frac{(\xi_1-\xi_2)U(\xi_1)V(\xi_2)}{(\xi_1-1)^{\frac{1}{2}}}$$

$$v(\xi_2) = 2\frac{R,_{\xi_2}}{R} + \frac{(1-2\xi_1\xi_2+\xi_2^2)}{(\xi_1-\xi_2)(1-\xi_2^2)} = \left\{\ln\frac{R^2(1-\xi_2^2)}{\xi_1-\xi_2}\right\},_{\xi_2}$$

$$R^2 = (\xi_1-\xi_2)\bar{U}(\xi_1)\bar{V}(\xi_2)$$

The simplest form for R is

$$U(\xi_1) = (\xi_1^2-1)^{\frac{1}{2}} \qquad V(\xi_2) = 1$$

$$R = (\xi_1-\xi_2)^{\frac{1}{2}} \quad R,_{\xi_1} = \frac{1}{2(\xi_1-\xi_2)^{\frac{1}{2}}} \quad R,_{\xi_1\xi_1} = \frac{-1}{4(\xi_1-\xi_2)^{\frac{3}{2}}}$$

$$R,_{\xi_2} = \frac{-1}{2(\xi_1-\xi_2)^{\frac{1}{2}}} \quad R,_{\xi_2\xi_2} = \frac{-1}{4(\xi_1-\xi_2)^{\frac{3}{2}}}$$

Laplace's Equation becomes

$$0 = (\xi_1^2-1)\left\{\frac{Y,_{\xi_1\xi_1}}{Y} + \frac{\xi_1}{\xi_1^2-1}\frac{Y,_{\xi_1}}{Y}\right\} + (1-\xi_2^2)\left\{\frac{Y,_{\xi_2\xi_2}}{Y} - \frac{\xi_2}{1-\xi_2^2}\frac{Y,_{\xi_2}}{Y}\right\}$$

$$+(\xi_1^2-1)\frac{-1}{4(\xi_1-\xi_2)^2}+\frac{(1-\xi_1\xi_2)}{(\xi_1-\xi_2)}\frac{1}{2(\xi_1-\xi_2)}+(1-\xi_2^2)\frac{-1}{4(\xi_1-\xi_2^2)}+\frac{(1-2\xi_1\xi_2+\xi_2^2)}{\xi_1-\xi_2}\frac{-1}{2(\xi_2-\xi_2)}+\frac{k_3^2}{(1-\xi_2^2)}$$

$$0=(\xi_1^2-1)\frac{Y_{,\xi_1\xi_1}}{Y}+\xi_1\frac{Y_{,\xi_1}}{Y}+(1-\xi_2^2)\frac{Y_{,\xi_2\xi_2}}{Y}-\xi_2\frac{Y_{,\xi_2}}{Y}+\frac{1}{4}+\frac{k_3^2}{(1-\xi_2^2)}$$

Let $Y(\xi_1,\xi_2)=X_1(\xi_1)X_2(\xi_2)$.

$$0=(\xi_1^2-1)\frac{X_{1,\xi_1\xi_1}}{X_1}+\xi_1\frac{X_{1,\xi_1}}{X_1}+(1-\xi_2^2)\frac{X_{2,\xi_2\xi_2}}{X_2}-\xi_2\frac{X_{2,\xi_2}}{X_2}+\frac{1}{4}+\frac{k_3^2}{(1-\xi_2^2)}$$

$$(\xi_1^2-1)\frac{X_{1,\xi_1\xi_1}}{X_1}+\xi_1\frac{X_{1,\xi_1}}{X_1}+\frac{1}{4}=k_1^2$$

$$(1-\xi_2^2)\frac{X_{2,\xi_2\xi_2}}{X_2}-\xi_2\frac{X_{2,\xi_2}}{X_2}+\frac{k_3^2}{(1-\xi_2^2)}=-k_1^2$$

$$(\xi_1^2-1)X_1''+\xi_1X_1'-(\frac{1}{4}+k_1^2)X_1=0$$

$$(1-\xi_2^2)^2X_2''-\xi_2(1-\xi_2^2)X_2'+\{k_3^2+k_1^2(1-\xi_2^2)\}X_2=0$$

The value for R given in the problem is incorrect; the proper value is given on page 665.

Problem 5.4

$$\xi = \ln(x^2 + y^2) - 2z$$

$$\eta = \frac{1}{2}(x^2 + y^2) + z$$

$$\phi = \tan^{-1} \frac{y}{x}$$

$$\left(\frac{\partial \xi_i}{\partial x_j}\right) = \begin{pmatrix} \frac{2x}{x^2+y^2} & \frac{2y}{x^2+y^2} & -2 \\ x & y & 1 \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix} = \frac{2x}{x^2+y^2}\left(\frac{-x}{x^2+y^2}\right) + \frac{2y}{x^2+y^2}\left(\frac{-y}{x^2+y^2}\right) - 2 = -2\frac{x^2+y^2+1}{x^2+y^2}$$

$$\left(\frac{\partial x_i}{\partial \xi_j}\right) = \begin{pmatrix} \frac{x}{2(x^2+y^2+1)} & \frac{x}{x^2+y^2+1} & -y \\ \frac{y}{2(x^2+y^2+1)} & \frac{y}{x^2+y^2+1} & x \\ \frac{-x^2-y^2}{2(x^2+y^2+1)} & \frac{1}{x^2+y^2+1} & 0 \end{pmatrix}$$

Problem 5.10

$$\nabla^2\psi = 0$$

$$g^{rr}\psi,_{rr} + g^{\phi\phi}\psi,_{\phi\phi} + \left(g^{rr},_r + g^{rr}\frac{g,_r}{2g}\right)\psi,_r + \left(g^{\phi\phi},_\phi + g^{\phi\phi}\frac{g,_\phi}{2g}\right)\psi,_\phi = 0$$

$$g^{rr} = 1 \qquad g^{\phi\phi} = \frac{1}{r^2} \qquad g = r^2$$

$$\psi,_{rr} + \frac{1}{r^2}\psi,_{\phi\phi} + \frac{1}{r}\psi,_r = 0$$

$$\left(r^2\psi,_{rr} + r\psi,_r\right) + \psi,_{\phi\phi} = 0$$

$$r^2\frac{X''}{X} + r\frac{X'}{X} + \frac{Y''}{Y} = 0$$

$$r^2\frac{X''}{X} + r\frac{X'}{X} - k^2 = 0 \qquad \frac{Y''}{Y} + k^2 = 0$$

$$Y'' = -k^2 Y$$

$$Y = Ae^{ik\phi} + Be^{-ik\phi}$$

$$Y(\phi + 2\pi) = Ae^{ik\phi}e^{i2\pi k} + B^{-ik\phi}e^{-i2\pi k}$$

$$Y(\phi + 2\pi) = Ae^{ik\phi} + Be^{-ik\phi}$$

$$e^{i2\pi k} = 1$$

$$k = \text{integer}$$

$$r^2\frac{X''}{X} + r\frac{X'}{X} - k^2 = 0$$

$$p(r) = \frac{1}{r} \qquad q(r) = -\frac{k^2}{r^2}$$

$r = 0$ is a simple pole of $p(r)$ and a second-order pole of $q(r)$. $r = 0$ is a regular singular point with $F = 1$ and $G = -k^2$. The indicial equation is

$$s^2 + (F-1)s + G = s^2 - k^2 = 0; \qquad s = \pm k.$$

Then

$$X = r^k u_1$$

$$X'' = k(k-1)r^{k-2}u_1 + 2kr^{k-1}u_1' + r^k u_1''$$

$$X' = kr^{k-1}u_1 + r^k u_1'$$

$$X'' + \frac{1}{r}X' - \frac{k^2}{r^2}X = k(k-1)r^{k-2}u_1 + 2kr^{k-1}u_1' + r^k u_1'' + kr^{k-2}u_1 + r^{k-1}u_1' - k^2 r^{k-2}u_1 = 0$$

$$0 = u_1'' + u_1'\left(\frac{2k+1}{r}\right)$$

Simplest solution: $u = c$; $X_1 = cr^k$.
Since $s_1 - s_2 = 2k$ =integer

$$X_2 = BX_1 \int e^{-\int p\,dr}\frac{dr}{X_1^2} = Br^k \int e^{-\int \frac{1}{r}\,dr}\frac{dr}{r^{2k}}$$

$$X_2 = Br^k \int e^{-\ln r}\frac{dr}{r^{2k}} = Br^k \int dr\, r^{-2k-1} = -\frac{B}{2k}r^{-k}$$

At the ordinary point $a \neq 0$, the basic set of solutions is

$$X_1 = 1 - \frac{1}{2}q(a)(r-a)^2 + \frac{1}{6}[q(a)p(a) - q'(a)](r-a)^3 + \dots$$

$$X_1 = 1 - \frac{1}{2}\left(-\frac{k^2}{a^2}\right)(r-a)^2 + \frac{1}{6}\left[-\frac{k^2}{a^2}\left(\frac{1}{a}\right) - \left(\frac{2k^2}{a^3}\right)\right](r-a)^3 + \dots$$

$$X_1 = 1 + \frac{k^2}{2a^2}(r-a)^2 - \frac{k^2}{2a^3}(r-a)^3 + \dots$$

$$X_2 = (r-a) - \frac{1}{2}p(a)(r-a)^2 + \frac{1}{6}[p^2(a) - p'(a) - q(a)](r-a)^3 + \dots$$

$$X_2 = (r-a) - \frac{1}{2a}(r-a)^2 + \frac{1}{6}\left[\frac{1}{a^2} - \left(-\frac{1}{a^2}\right) + \frac{k^2}{a^2}\right](r-a)^3 + \dots$$

$$X_2 = (r-a) - \frac{1}{2a}(r-a)^2 + \frac{2+k^2}{6a^2}(r-a)^3 + \dots$$

For $Y'' + k^2Y = 0$, $p(\phi) = 0$ and $q(\phi) = k^2$ and

$$Y_1 = 1 - \frac{1}{2}k^2\phi^2 + \frac{1}{6}[0]\phi^3 + \dots$$

$$Y_2 = \phi - \frac{k^2}{6}\phi^3 + \dots$$

Problem 5.11

$$\psi'' + (k - x^2)\psi = 0 \quad p(x) = 0 \quad q(x) = k - x^2$$

At $x = 0$:

$$y_1 = 1 - \frac{1}{2}q(0)x^2 + \frac{1}{6}\left[q(0)p(0) - q'(0)\right]x^3 + \dots$$

$$y_1 = 1 - \frac{1}{2}kx^2 + \frac{1}{6}\left[k \cdot 0 - (-2 \cdot 0)\right]x^3 + \dots$$

$$y_1 = 1 - \frac{1}{2}kx^2 + \dots$$

$$y_2 = x - \frac{1}{2}p(0)x^2 + \frac{1}{6}\left[p^2(0) - p'(0) - q(0)\right]x^3 + \dots$$

$$y_2 = x - \frac{k}{6}x^3 + \dots$$

Problem 5.12

The homogeneous equation

$$\psi'' - \frac{6}{x^2}\psi = 0$$

has a regular singular point at $x = 0$: $F(x) = 0$; $G(x) = 6$. The indicial equation is

$$s^2 + (F(0) - 1)\, s + G(0) = (s-3)(s+2) = 0$$

$$s_1 = 3; \quad s_2 = -2$$

$$X_1 = x^{-2}u_1$$

$$X_1' = -2x^{-3}u_1 + x^{-2}u_1'$$

$$X_1'' = 6x^{-4}u_1 - 4x^{-3}u_1' + x^{-2}u_1''$$

$$6x^{-4}u_1 - 4x^{-3}u_1' + x^{-2}u_1'' - \frac{6}{x^2}(x^{-2}u_1) = 0$$

$$u'' = \frac{4}{x}u_1'$$

$$\ln u_1' = 4\ln x; \quad u_1' = x^4; \quad u_1 = \frac{1}{5}x^5$$

The simplest solution is $u = c$.

$$X_1 = c_1 x^{-2}$$

$$X_2 = BX_1 \int e^{-\int p\, dx} \frac{dx}{X_1^2}$$

$$X_2 = Bc_1 x^{-2} \int \frac{dx}{c_1^2 x^{-4}} = \frac{B}{c_1}x^{-2} \int x^4\, dx$$

$$X_2 = \frac{B}{c_1}x^{-2}\left(\frac{x^5}{5}\right) = \frac{B}{5c_1}x^3 = c_2 x^3$$

$$\psi - c_1 x^{-2} + c_2 x^3$$

The inhomogeneous equation

$$\psi'' - \frac{6}{x^2}\psi = x \ln x$$

is solved by Eq. 5.2.19, p. 530:

$$\psi = X_1\left[c_1 - \int \frac{r(x) X_2\, dx}{\Delta}\right] + X_2\left[c_2 + \int \frac{r(x) X_1\, dx}{\Delta}\right]$$

with

$$r(x) = x \ln x$$

$$\Delta = (X_1 X_2' - X_1' X_2) = \left\{x^{-2}(3x^2) - x^3(-2x^{-3})\right\} = 5$$

$$\psi = x^{-2}\left[c_1 - \int \frac{(x\ln x)(x^3)\,dx}{5}\right] + x^3\left[c_2 + \int \frac{(x\ln x)(x^{-2})\,dx}{5}\right]$$

$$\psi = c_1 x^{-2} + c_2 x^3 - \frac{1}{5}x^{-2}\int x^4 \ln x\,dx + \frac{1}{5}x^3 \int \frac{\ln x\,dx}{x}$$

$$\psi = c_1 x^{-2} + c_2 x^3 - \frac{1}{5}x^{-2}\left\{\frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^4\,dx\right\} + \frac{1}{5}x^3 \int \ln x\,d(\ln x)$$

$$\psi = c_1 x^{-2} + c_2 x^3 - \frac{1}{5}x^{-2}\left\{\frac{x^5 \ln x}{5} - \frac{x^5}{25}\right\} + \frac{1}{5}x^3\left\{\frac{(\ln x)^2}{2}\right\}$$

$$\psi = c_1 x^{-2} + c_2 x^3 - \frac{x^3 \ln x}{25} + \frac{x^3}{125} + \frac{x^3(\ln x)^2}{10}$$

Problem 5.18

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E-V)\psi = 0$$

$$\psi'' + \frac{2m}{\hbar^2}\left(E + \frac{\hbar^2 A^2}{2m}\operatorname{sech}^2\frac{x}{d}\right)\psi = 0$$

$$\psi'' + \left(\frac{2mE + \hbar^2 A^2}{\hbar^2} - A^2\tanh^2\frac{x}{d}\right)\psi = 0$$

$$z = \frac{1}{2} + \frac{1}{2}\tanh\frac{x}{d}$$

$$\frac{dz}{dx} = \frac{1}{2d\cosh^2\frac{x}{d}} = \frac{1}{2d}\left(1-\tanh^2\frac{x}{d}\right) = \frac{1}{2d}\left(1-(2z-1)^2\right) = \frac{2z(1-z)}{d}$$

$$\frac{d^2z}{dx^2} = -\frac{1}{d^2}\frac{\sinh\frac{x}{d}}{\cosh^3\frac{x}{d}} = \frac{1}{d^2}\tanh\frac{x}{d}\left(\tanh^2\frac{x}{d}-1\right) = \frac{1}{d^2}(2z-1)\left((2z-1)^2-1\right) = \frac{4z(z-1)(2z-1)}{d^2}$$

$$\frac{d^2\psi}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{d\psi}{dz}\frac{d^2z}{dx^2} + \left(\frac{2mE}{\hbar^2} + 4A^2 z(z-1)\right)\psi = 0$$

$$\frac{d^2\psi}{dz^2}\frac{4z^2(1-z)^2}{d^2} + \frac{d\psi}{dz}\frac{4z(z-1)(2z-1)}{d^2} + \left(\frac{2mE}{\hbar^2} - 4A^2 z(z-1)\right)\psi = 0$$

$$\frac{d^2\psi}{dx^2} + \frac{2z-1}{z(z-1)}\frac{d\psi}{dx} + \left(\frac{2mEd^2 - 4\hbar^2 A^2 d^2 z(z-1)}{4\hbar^2 z^2(z-1)^2}\right)\psi = 0$$

This equation has two regular singular points located at $z=0$ and $z=1$. For the singularity at $z=0$,

$$F(z) = \frac{2z-1}{z-1} \quad G(z) = \frac{2mEd^2 - 4\hbar^2 A^2 d^2 z(z-1)}{4\hbar^2(z-1)^2}$$

$$F(0) = 1 \quad G(0) = \frac{mEd^2}{2\hbar^2} \quad s^2 + \frac{mEd^2}{2\hbar^2} = 0$$

$$s = \pm\frac{d}{\hbar}\sqrt{\frac{mE}{2}}$$

For the singularity at $z=1$,

$$F(z) = \frac{2z-1}{z} \quad G(z) = \frac{2mEd^2 - 4\hbar^2 A^2 d^2 z(z-1)}{4\hbar^2 z^2}$$

$$F(1) = 1 \quad G(1) = \frac{mEd^2}{2\hbar^2} \quad s^2 + \frac{mEd^2}{2\hbar^2} = 0$$

$$s = \pm\frac{d}{\hbar}\sqrt{\frac{mE}{2}}$$

For the point at infinity,

$$P(w) = \frac{1}{w-1} = \frac{w(w-1)}{w} \qquad Q(w) = \frac{2mEd^2w^2 + 4\hbar^2A^2d^2(1-w)}{4\hbar^2w^2(1-w)^2}$$

and $w = 0$ is a regular singular point. Let

$$\psi = \{z(1-z)\}^{\frac{d}{\hbar}\sqrt{\frac{mE}{2}}}F.$$

$$f = \{z(1-z)\}^{\frac{d}{\hbar}\sqrt{\frac{mE}{2}}}$$

$$\frac{f'}{f} = -\frac{d}{\hbar}\sqrt{\frac{mE}{2}}\frac{1-2z}{z(z-1)} \qquad \frac{f''}{f} = \left(\frac{f'}{f}\right)' + \left(\frac{f'}{f}\right)^2 = -\frac{2d}{\hbar}\sqrt{\frac{mE}{2}}\frac{2z^2-2z+1}{z^2(z-1)^2} + \frac{d^2mE}{2\hbar^2}\frac{(2z-1)^2}{z^2(z-1)^2}$$

$$\frac{\psi'}{\psi} = \frac{d}{\hbar}\sqrt{\frac{mE}{2}}\frac{2z-1}{z(z-1)} + \frac{F'}{F} \qquad \frac{\psi''}{\psi} = -\frac{2d}{\hbar}\sqrt{\frac{mE}{2}}\frac{1}{(z-1)^2} + \frac{2d}{\hbar}\sqrt{\frac{mE}{2}}\frac{2z-1}{z(z-1)}\frac{F'}{F} + \frac{F''}{F}$$

$$\frac{\psi''}{\psi} + \frac{2z-1}{z(z-1)}\frac{\psi'}{\psi} + \frac{2mEd^2 - 4\hbar^2A^2d^2z(z-1)}{4\hbar^2z^2(z-1)^2} = 0$$

$$\frac{F''}{F} + \frac{F'}{F}\left\{\frac{2d}{\hbar}\sqrt{\frac{mE}{2}}\frac{2z-1}{z(z-1)} + \frac{2z-1}{z(z-1)}\right\} + \frac{2mEd^2 - 4\hbar^2A^2d^2z(z-1)}{4\hbar^2z^2(z-1)^2} - \frac{2d}{\hbar}\sqrt{\frac{mE}{2}}\frac{1}{(z-1)^2}$$

$$+\frac{2d}{\hbar}\sqrt{\frac{mE}{2}}\frac{2z-1}{z(z-1)} + \frac{d}{\hbar}\sqrt{\frac{mE}{2}}\frac{(2z-1)^2}{z^2(z-1)^2}$$

Problem 5.20

$$\nabla^2\psi + k^2\psi = 0$$

$$g^{\alpha\beta}\psi,_{\alpha\beta} + \left(g^{\alpha\beta},_\beta + g^{\alpha\beta}\frac{g,_\beta}{g}\right)\psi,_\alpha + k^2\psi = 0$$

$$g^{rr} = 1 \qquad g^{\theta\theta} = \frac{1}{r^2} \qquad g^{\phi\phi} = \frac{1}{r^2\sin^2\theta} \qquad g = r^2\sin\theta$$

$$\psi,_{rr} + \frac{2}{r}\psi,_r + \frac{1}{r^2}\psi,_{\theta\theta} + \frac{1}{r^2}\cot\theta\psi,_\theta + \frac{1}{r^2\sin^2\theta}\psi,_{\phi\phi} + k^2\psi = 0$$

Let $\psi = \psi(r)T(\theta)\Phi(\phi)$.

$$\frac{\psi''}{\psi} + \frac{2}{r}\frac{\psi'}{\psi} + \frac{1}{r^2}\frac{T''}{T} + \frac{1}{r^2}\cot\theta\frac{T'}{T} + \frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi} + k^2 = 0$$

$$\sin^2\theta\left\{r^2\frac{\psi''}{\psi} + 2r\frac{\psi'}{\psi} + r^2k^2 + \frac{T''}{T} + \cot\theta\frac{T'}{T}\right\} - m^2 = 0 \qquad \frac{\Phi''}{\Phi} + m^2 = 0$$

$$\frac{\Phi''}{\Phi} = -m^2 \qquad \Phi = e^{im\phi} + e^{-im\phi}$$

$$\sin^2\theta\left\{r^2\frac{\psi''}{\psi} + 2r\frac{\psi'}{\psi} + r^2k^2 + \frac{T''}{T} + \cot\theta\frac{T'}{T}\right\} - m^2 = 0$$

$$r^2\frac{\psi''}{\psi} + 2r\frac{\psi'}{\psi} + r^2k^2 - n(n+1) = 0 \qquad \frac{T''}{T} + \cot\theta\frac{T'}{T} + n(n+1) - \frac{m^2}{\sin^2\theta} = 0$$

$$T'' + \cot\theta T' + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}T = 0$$

Let $u = \cos\theta$.

$$T'' - \frac{2u}{1-u^2}T' + \{n(n+1)(1-u^2) - m^2\}\frac{T}{(1-u^2)^2} = 0$$

This is the associated Legendre equation with solutions which are polynomial when n is an integer.

$$T = P_m^n(u) + Q_m^n(u)$$

$$r^2\frac{\psi''}{\psi} + 2r\frac{\psi'}{\psi} + r^2k^2 - n(n+1) = 0$$

Let $x = kr$.

$$\psi'' + \frac{2}{x}\psi' + 1 - \frac{n(n+1)}{x^2} = 0$$

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\psi}{dx}\right) + \left[1 - \frac{n(n+1)}{x^2}\right]\psi = 0$$

Let $\psi = fJ$.

$$J'' + \left[\frac{2f'}{f} + \frac{2}{x}\right] J' + \left[1 - \frac{n(n+1)}{x^2} + \frac{1}{x}\frac{2f'}{f} + \frac{f''}{f}\right] J = 0$$

Set

$$\frac{2f'}{f} + \frac{2}{x} = \frac{1}{x}$$

so that

$$\frac{f'}{f} = -\frac{1}{2x} \qquad f = x^{-\frac{1}{2}} \qquad \frac{f''}{f} = \left(\frac{f'}{f}\right)' + \left(\frac{f'}{f}\right)^2 = \frac{3}{4x^2}$$

$$J'' + \frac{1}{x}J' + \left[1 - \frac{(n+\frac{1}{4})^2}{x^2}\right] J = 0$$

This is Bessel's equation with solutions

$$\sqrt{\frac{\pi}{2}} J_{n+\frac{1}{2}}(x); \qquad \sqrt{\frac{\pi}{2}} N_{n+\frac{1}{2}}(x).$$

$$\psi_n = fJ = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x), \qquad \sqrt{\frac{\pi}{2x}} N_{n+\frac{1}{2}}(x)$$

Set $\psi = fF = x^s e^{-tx} F$.

$$\frac{f'}{f} = \frac{s}{x} - t \qquad \frac{f''}{f} = \frac{s(s-1)}{x^2} - \frac{2st}{x} + t^2$$

$$\frac{\psi'}{\psi} = \frac{s}{x} - t + \frac{F'}{F} \qquad \frac{\psi''}{\psi} = \frac{s(s-1)}{x^2} - \frac{2st}{x} + t^2 + 2\left(\frac{s}{x} - t\right)\frac{F'}{F} + \frac{F''}{F}$$

$$\frac{\psi''}{\psi} + \frac{2}{x}\frac{\psi'}{\psi} + 1 - \frac{n(n+1)}{x^2} = 0$$

$$\frac{F''}{F} + \frac{F'}{F}\left\{\frac{2(s+1)}{x} - 2t\right\} + 1 + t^2 - \frac{2(s+1)t}{x} - \frac{n(n+1) - s(s+1)}{x^2} = 0$$

This is the confluent hypergeometric equation

$$zF'' + (c - z)F' - aF = 0$$

if

$$t = -i \qquad s = n \qquad z = 2ix \qquad c = -2(n+1) \qquad a = -n-1$$

so that

$$\psi_n = x^n e^{ix} F(-n-1| -2n-2|2ix).$$

Or

$$t = -i \qquad s = n-1 \qquad z = 2ix \qquad c = -2n \qquad a = -n$$

so that

$$\psi_n = x^{n-1} e^{ix} F(-n| -2n|2ix).$$

The proposed solution given in the book is in error.

Problem 5.21

$$\nabla^2\psi + \left(\frac{2mE}{\hbar^2} + \frac{2me^2Z}{\hbar^2 r}\right)\psi = 0$$

$$g^{\alpha\beta}\psi_{,\alpha\beta} + \left(g^{\alpha\beta}{}_{,\beta} + g^{\alpha\beta}\frac{g_{,\beta}}{g}\right)\psi_{,\alpha} + \left(\frac{2mE}{\hbar^2} + \frac{2me^2Z}{\hbar^2 r}\right)\psi = 0$$

$$g^{rr} = 1 \qquad g^{\theta\theta} = \frac{1}{r^2} \qquad g^{\phi\phi} = \frac{1}{r^2\sin^2\theta} \qquad g = r^2\sin\theta$$

$$\psi_{,rr} + \frac{2}{r}\psi_{,r} + \frac{1}{r^2}\psi_{,\theta\theta} + \frac{1}{r^2}\cot\theta\psi_{,\theta} + \frac{1}{r^2\sin^2\theta}\psi_{,\phi\phi} + \left(\frac{2mE}{\hbar^2} + \frac{2me^2Z}{\hbar^2 r}\right)\psi = 0$$

Let $\psi = \psi(r)T(\theta)\Phi(\phi)$.

$$\frac{\psi''}{\psi} + \frac{2}{r}\frac{\psi'}{\psi} + \frac{1}{r^2}\frac{T''}{T} + \frac{1}{r^2}\cot\theta\frac{T'}{T} + \frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi} + \left(\frac{2mE}{\hbar^2} + \frac{2me^2Z}{\hbar^2 r}\right) = 0$$

$$\sin^2\theta\left\{r^2\frac{\psi''}{\psi} + 2r\frac{\psi'}{\psi} + r^2\left(\frac{2mE}{\hbar^2} + \frac{2me^2Z}{\hbar^2 r}\right) + \frac{T''}{T} + \cot\theta\frac{T'}{T}\right\} - m^2 = 0 \qquad \frac{\Phi''}{\Phi} + m^2 = 0$$

$$\frac{\Phi''}{\Phi} = -m^2 \qquad \Phi = e^{im\phi} + e^{-im\phi}$$

$$\sin^2\theta\left\{r^2\frac{\psi''}{\psi} + 2r\frac{\psi'}{\psi} + r^2\left(\frac{2mE}{\hbar^2} + \frac{2me^2Z}{\hbar^2 r}\right) + \frac{T''}{T} + \cot\theta\frac{T'}{T}\right\} - m^2 = 0$$

$$r^2\frac{\psi''}{\psi} + 2r\frac{\psi'}{\psi} + r^2\left(\frac{2mE}{\hbar^2} + \frac{2me^2Z}{\hbar^2 r}\right) - n(n+1) = 0 \qquad \frac{T''}{T} + \cot\theta\frac{T'}{T} + n(n+1) - \frac{m^2}{\sin^2\theta} = 0$$

$$T'' + \cot\theta T' + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}T = 0$$

Let $u = \cos\theta$.

$$T'' - \frac{2u}{1-u^2}T' + \{n(n+1)(1-u^2) - m^2\}\frac{T}{(1-u^2)^2} = 0$$

This is the associated Legendre equation with solutions which are polynomial when n is an integer.

$$T = P_m^n(u) + Q_m^n(u)$$

$$\frac{\psi''}{\psi} + \frac{2}{r}\frac{\psi'}{\psi} + \frac{2mE}{\hbar^2} + \frac{2me^2Z}{\hbar^2 r} - \frac{n(n+1)}{r^2} = 0$$

Set $\psi = fF = r^s e^{-tr}F$.

$$\frac{f'}{f} = \frac{s}{r} - t \qquad \frac{f''}{f} = \frac{s(s-1)}{r^2} - \frac{2st}{r} + t^2$$

$$\frac{\psi'}{\psi} = \frac{s}{r} - t + \frac{F'}{F} \qquad \frac{\psi''}{\psi} = \frac{s(s-1)}{r^2} - \frac{2st}{r} + t^2 + 2\left(\frac{s}{r} - t\right)\frac{F'}{F} + \frac{F''}{F}$$

$$\frac{\psi''}{\psi} + \frac{2}{r}\frac{\psi'}{\psi} + \frac{2mE}{\hbar^2} + \frac{2me^2Z}{\hbar^2 r} - \frac{n(n+1)}{r^2} = 0$$

$$\frac{F''}{F} + \frac{F'}{F}\left\{\frac{2(s+1)}{r} - 2t\right\} + \frac{2mE}{\hbar^2} + t^2 - \frac{2(s+1)t}{r} + \frac{2me^2Z}{\hbar^2 r} - \frac{n(n+1) - s(s+1)}{r^2} = 0$$

This is the confluent hypergeometric equation

$$zF'' + (c-z)F' - aF = 0$$

if

$$t = k = \frac{\sqrt{-2mE}}{\hbar} \qquad s = n \qquad z = 2kr \qquad c = 2(n+1) \qquad a = n + 1 - \frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}}$$

so that

$$\psi_n = r^n e^{-r\sqrt{\frac{-2mE}{\hbar^2}}} F\left(n + 1 - \frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}}\,|2(n+1)|\,r\sqrt{\frac{-2mE}{\hbar^2}}\right).$$

Independence of θ and ϕ means $n = 0$. The two asymptotic series for this function are

$$F_1 = Az^{-n-1+\frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}}}\left[1 + \frac{\left(1 - \frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}}\right)\left(-\frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}}\right)}{z} + \right.$$

$$\left.\frac{\left(1 - \frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}}\right)\left(2 - \frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}}\right)\left(-\frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}}\right)\left(1 - \frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}}\right)}{2!z^2} + \dots\right]$$

and

$$F_2 = Az^{-1-\frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}}}\left[1 + \frac{\left(\frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}}\right)\left(1 + \frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}}\right)}{z} + \right.$$

$$\left.\frac{\left(\frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}}\right)\left(1 + \frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}}\right)\left(1 + \frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}}\right)\left(2 + \frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}}\right)}{2!z^2} + \dots\right]$$

The two series truncate when

$$n' \pm \frac{e^2Z}{\hbar}\sqrt{\frac{m}{-2E}} = 0 \qquad - E = \frac{2\hbar^2 n'^2}{me^4Z^2}.$$

The solution is finite for all values of r when $E \leq 0$.

Problem 5.23

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E-V)\psi = 0$$

$$V = \frac{\hbar^2}{2m}U_0\left[\cos^2\frac{\pi x}{l_x} + \cos^2\frac{\pi y}{l_y} + \cos^2\frac{\pi z}{l_z}\right]$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \left(\frac{2mE}{\hbar^2} - U_0\left[\cos^2\frac{\pi x}{l_x} + \cos^2\frac{\pi y}{l_y} + \cos^2\frac{\pi z}{l_z}\right]\right)\psi = 0$$

$$\left\{\frac{\partial^2\psi}{\partial x^2} - U_0\cos^2\frac{\pi x}{l_x}\psi\right\} + \left\{\frac{\partial^2\psi}{\partial y^2} - U_0\cos^2\frac{\pi y}{l_y}\psi\right\} + \left\{\frac{\partial^2\psi}{\partial z^2} - U_0\cos^2\frac{\pi z}{l_z}\psi\right\} = -\frac{2mE}{\hbar^2}\psi$$

$$\psi(x,y,z) = X(x)Y(y)Z(z)$$

$$\left\{\frac{X''}{X} - U_0\cos^2\frac{\pi x}{l_x}\right\} + \left\{\frac{Y''}{Y} - U_0\cos^2\frac{\pi y}{l_y}\right\} + \left\{\frac{ZY\prime\prime}{Z} - U_0\cos^2\frac{\pi z}{l_z}\right\} = -\frac{2mE}{\hbar^2}$$

This equation is separable into three parts, each of which is of the form

$$\frac{X_i{}''}{X_i} + \frac{2m\alpha_{\xi_i}E}{\hbar^2} - U_0\cos^2\frac{\pi\xi_i}{l_{\xi_i}} = 0$$

where

$$\alpha_x + \alpha_y + \alpha_z = 1.$$

This is Mathieu's equation

$$\frac{\partial^2 X_i}{\partial\phi^2} + (b - h^2\cos^2\phi)X_i = 0$$

with

$$\phi = \frac{\pi\xi_i}{l_{\xi_i}} \qquad b = \frac{2m\alpha_{\xi_i}{l_{\xi_i}}^2E}{\pi^2\hbar^2} \qquad h^2 = \frac{l_{\xi_i}^2U_0}{\pi^2}.$$

Problem 5.30

$$z\frac{d^3\psi}{dz^3} - \psi = 0$$

$$f(t) = t^3 \qquad F(t) = -1 \qquad \int \frac{F(t)}{f(t)}\,dt = \int -\frac{1}{t^3}\,dt = \frac{1}{2t^2}$$

$$\psi = \int e^{zt+\frac{1}{2t^2}}\,\frac{dt}{t^3} \qquad t = \frac{i}{u} \qquad dt = -\frac{i}{u^2}\,du$$

$$\psi = \int e^{\frac{iz}{u}}\,e^{-\frac{1}{2}u^2}\,u\,du$$

$$P(u)|_{-\infty}^{\infty} = e^{\frac{iz}{u}}\,e^{-\frac{1}{2}u^2}|_{-\infty}^{\infty} = 0$$

$$\psi = \int_{-\infty}^{\infty} e^{\frac{iz}{u}}\,e^{-\frac{1}{2}u^2}\,u\,du = \int_{-\infty}^{\infty} \cos\left(\frac{z}{u}\right)e^{-\frac{1}{2}u^2}\,u\,du + i\int_{-\infty}^{\infty} \sin\left(\frac{z}{u}\right)e^{-\frac{1}{2}u^2}\,u\,du$$

The first integral is odd and therefore $\rightarrow 0$. The second integral is even and

$$\psi = 2i\int_{-\infty}^{\infty} \sin\left(\frac{z}{u}\right)e^{-\frac{1}{2}u^2}\,u\,du.$$

If it is true for all $k \le n$, then it is true for $n+1$:

$$Q_n(z) = \frac{1}{2}P_n(z)\ln\left(\frac{z+1}{z-1}\right) - W_{n-1}(z)$$
$$Q_{n-1}(z) = \frac{1}{2}P_{n-1}(z)\ln\left(\frac{z+1}{z-1}\right) - W_{n-2}(z)$$

From the recurrence relation

$$(n+1)p_{n+1}(z) - (2n+1)zp_n(z) + np_{n-1}(z) = 0$$

obtain

$$\begin{aligned}
(n+1)Q_{n+1}(z) &= (2n+1)zQ_n(z) - nQ_{n-1}(z) \\
&= (2n+1)z\left\{\frac{1}{2}P_n(z)\ln\left(\frac{z+1}{z-1}\right) - W_{n-1}(z)\right\} - n\left\{\frac{1}{2}P_{n-1}(z)\ln\left(\frac{z+1}{z-1}\right) - W_{n-2}(z)\right\} \\
&= \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)\{(2n+1)zP_n(z) - nP_{n-1}(z)\} - (2n+1)zW_{n-1}(z) + nW_{n-2}(z) \\
&= \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)(n+1)P_{n+1}(z) - (2n+1)zW_{n-1}(z) + nW_{n-2}(z) \\
Q_{n+1}(z) &= \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)P_{n+1}(z) - \frac{2n+1}{n+1}zW_{n-1}(z) + \frac{n}{n+1}W_{n-2}(z)
\end{aligned}$$

$$\begin{aligned}
\frac{2n+1}{n+1}zW_{n-1}(z) - \frac{n}{n+1}W_{n-2}(z) &= \frac{2n+1}{n+1}z\left\{a_{n-1}z^{n-1} + \ldots + a_0\right\} - \frac{n}{n+1}\left\{b_{n-2}z^{n-2}\ldots + b_0\right\} \\
\frac{2n+1}{n+1}zW_{n-1}(z) - \frac{n}{n+1}W_{n-2}(z) &= \frac{2n+1}{n+1}a_{n-1}z^n + \frac{2n+1}{n+1}a_{n-2}z^{n-1} + \left\{\frac{(2n+1)a_{n-3} - nb_{n-2}}{n+1}\right\}z^{n-2} + \ldots + b_0 \\
&= W_n(z)
\end{aligned}$$

It is true for $n = 0$:

$$Q_0(z)P_0'(z) - P_0(z)Q_0'(z) = \frac{1}{1-z^2}$$

$$P_0 = 1; \qquad P_0'(z) = 0$$

$$-Q_0'(z) = \frac{1}{z^2-1}$$

$$Q_0(z) = \int_z^\infty \frac{du}{u^2-1} = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right) = \frac{1}{2}P_0(z)\ln\left(\frac{z+1}{z-1}\right)$$

It is true for $n = 1$: From the recurrence relation

$$p'(z)n + 1 - zp'(z)_n = (n+1)p_n(z)$$

$$\begin{aligned}
P_1'(z) - zP_0'(z) &= P_0(z) \\
P_1'(z) &= 1 + z(0) \\
P_1(z) &= \int_0^z du = z \\
Q_1(z)P_1'(z) - P_1(z)Q_1'(z) &= \frac{1}{1-z^2}
\end{aligned}$$

$$
\begin{aligned}
Q_1(z) - zQ_1'(z) &= \frac{1}{1-z^2} \\
\frac{Q_1(z) - zQ_1'(z)}{z^2} &= \frac{1}{z^2(1-z^2)} \\
\frac{d}{dz}\left(\frac{Q_1(z)}{z}\right) &= \frac{1}{z^2(z^2-1)} = \frac{1-z^2+z^2}{z^2(z^2-1)} \\
&= -\frac{1}{z^2} + \frac{1}{z^2-1} = \frac{d}{dz}\left\{\frac{1}{z} + \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)\right\} \\
Q_1(z) &= \frac{1}{2}P_1(z)\ln\left(\frac{z+1}{z-1}\right) + 1 \\
&= \frac{1}{2}P_1(z)\ln\left(\frac{z+1}{z-1}\right) + W_0(z)
\end{aligned}
$$

From Eq. 5.3.29, page 597, z cannot be a real number between -1 and $+1$. $\ln\left(\frac{z+1}{z-1}\right)$ has branch points at these two values so that the line connecting them is a branch cut. The difference across the cut is $2\pi i$.

$$
Q_n(x+i\epsilon) = \frac{1}{2}P_n(x+i\epsilon)\ln\left(\frac{x+1+i\epsilon}{x-1+i\epsilon}\right) - W_{n-1}(x+i\epsilon)
$$

$$
Q_n(x-i\epsilon) = \frac{1}{2}P_n(x+i\epsilon)\left\{\ln\left(\frac{x+1+i\epsilon}{x-1+i\epsilon}\right) + 2\pi i\right\} - W_{n-1}(x+i\epsilon)
$$

Expanding each term on the right side in a Launrent series,

$$
\begin{aligned}
Q_n(x+i\epsilon) &= \frac{1}{2}\left\{P_n(x) + P_n'(x) + \ldots\right\}\left[\ln\left(\frac{x+1}{x-1}\right) + \frac{i\epsilon}{x^2-1} + \ldots\right] - W_{n-1}(x) - i\epsilon W_{n-1}'(x) + \ldots \\
&= \frac{1}{2}P_n(x)\ln\left(\frac{x+1}{x-1}\right) - W_{n-1}(x) + i\epsilon\left\{\frac{1}{2}P_n'(x)\ln\left(\frac{x+1}{x-1}\right) + \frac{1}{2}\frac{P_n(x)}{x^2-1} - W_{n-1}'(x)\right\} + \ldots \\
Q_n(x-i\epsilon) &= \frac{1}{2}\left\{P_n(x) + P_n'(x) + \ldots\right\}\left[\ln\left(\frac{x+1}{x-1}\right) + 2\pi i - \frac{i\epsilon}{x^2-1} + \ldots\right] - W_{n-1}(x) + i\epsilon W_{n-1}'(x) + \ldots \\
&= \frac{1}{2}P_n(x)\left\{\ln\left(\frac{x+1}{x-1}\right) + 2\pi i\right\} - W_{n-1}(x) - i\epsilon\left[\frac{1}{2}P_n'(x)\left\{\ln\left(\frac{x+1}{x-1}\right) + 2\pi i\right\} + \right. \\
&\qquad \left. \frac{1}{2}\frac{P_n(x)}{x^2-1} - W_{n-1}'(x)\right] + \ldots
\end{aligned}
$$

$$
Q_n(x+i\epsilon) - Q_n(x-i\epsilon) = -\pi i P_n(x) + i\epsilon\left\{P_n'(x)\ln\left(\frac{x+1}{x-1}\right) + 2\pi i + \frac{P_n(x)}{x^2-1} - 2W_{n-1}'(x)\right\} + \ldots
$$

$$
\lim_{\epsilon\to 0}\left\{Q_n(x+i\epsilon) - Q_n(x-i\epsilon)\right\} = -\pi i P_n(x)
$$

Problem 5.34

Problem 5.37

From Equation 5.3.65, page 620,

$$\begin{aligned}
J_m(z) &= \frac{e^{-i\pi m/2}}{\pi} \int_0^{\pi} e^{iz\cos\phi} \cos(m\phi)\, d\phi \\
&= \frac{e^{-i\pi m/2}}{\pi} \int_0^{-\pi} e^{iz\cos(-u)} \cos(-mu)\, d(-u) \\
&= \frac{e^{-i\pi m/2}}{\pi} \int_{-\pi}^{0} e^{iz\cos u} \cos(mu)\, du \\
2J_m(z) &= \frac{e^{-i\pi m/2}}{\pi} \left\{ \int_{-\pi}^{0} e^{iz\cos u} \cos(mu)\, du + \int_0^{\pi} e^{iz\cos\phi} \cos(m\phi)\, d\phi \right\} \\
J_m(z) &= \frac{(e^{-i\pi/2})^m}{2\pi} \int_0^{2\pi} e^{iz\cos u} \cos(mu)\, du \\
&= \frac{(-i)^m}{2\pi} \int_0^{2\pi} e^{iz\cos u} \cos(mu)\, du \\
&= \frac{1}{2\pi i^m} \int_0^{2\pi} e^{iz\cos u} \cos(mu)\, du
\end{aligned}$$

Problem 5.38

$$\begin{aligned}
U_2(a|c|z) &= \frac{e^{ia\pi}z^{-a}}{\Gamma(a)}\int_0^\infty e^{-u}u^{a-1}\left(1+\frac{u}{z}\right)^{c-a-1}du \\
&= \frac{e^{ia\pi}z^{-a}}{\Gamma(a)}\int_0^\infty e^{-u}u^{a-1}\sum_{m=0}^\infty \frac{\Gamma(c-a)}{\Gamma(m+1)\Gamma(c-a-m)}\left(-\frac{u}{z}\right)^m du \\
&= \frac{e^{ia\pi}z^{-a}\Gamma(c-a)}{\Gamma(a)}\sum_{m=0}^\infty \frac{z^{-m}}{\Gamma(m+1)\Gamma(c-a-m)}\int_0^\infty e^{-u}u^{a+m-1}du \\
&= \frac{e^{ia\pi}z^{-a}\Gamma(c-a)}{\Gamma(a)}\sum_{m=0}^\infty \frac{z^{-m}\Gamma(a+m)}{\Gamma(m+1)\Gamma(c-a-m)}
\end{aligned}$$

Equation 5.3.3, page 579:

$$2\pi i\sum_{m=0}^\infty \frac{G(m)z^m}{m!} = \int_{-i\,\infty}^{i\,\infty} G(t)\Gamma(-t)(-z)^t\,dt$$

$$\begin{aligned}
U_2(a|c|z) &= \frac{e^{ia\pi}z^{-a}\Gamma(c-a)}{2\pi i\Gamma(a)}\left\{2\pi i\sum_{m=0}^\infty \frac{\Gamma(a+m)}{\Gamma(c-a-m)}\frac{(\frac{1}{z})^m}{m!}\right\} \\
&= \frac{e^{ia\pi}z^{-a}\Gamma(c-a)}{2\pi i\Gamma(a)}\int_{-i\,\infty}^{i\,\infty}\frac{\Gamma(a+t)}{\Gamma(c-a-t)}\Gamma(-t)(-z)^{-t}\,dt \\
&= \frac{e^{ia\pi}z^{-a}}{2\pi i\Gamma(a)}\int_{-i\,\infty}^{i\,\infty}\frac{\Gamma(c-a)\Gamma(a+t)}{\Gamma(c-a-t)}\Gamma(-t)(-z)^{-t}\,dt \\
&= \frac{e^{ia\pi}z^{-a}}{2\pi i\Gamma(a)}\int_{-i\,\infty}^{i\,\infty}\frac{(c-a-1)(c-a-2)\ldots(c-a-t)\Gamma(c-a-t)\Gamma(a+t)}{\Gamma(c-a-t)}\Gamma(-t)(-z)^{-t}\,dt \\
&= \frac{e^{ia\pi}z^{-a}}{2\pi i}\int_{-i\,\infty}^{i\,\infty}\frac{(1+a-c)(2+a-c)\ldots(t+a-c)\Gamma(a+t)}{\Gamma(a)}\Gamma(-t)z^{-t}\,dt \\
&= \frac{e^{ia\pi}z^{-a}}{2\pi i}\int_{-i\,\infty}^{i\,\infty}\frac{\Gamma(1+a-c)(1+a-c)(2+a-c)\ldots(t+a-c)\Gamma(a+t)}{\Gamma(1+a-c)\Gamma(a)}\Gamma(-t)z^{-t}\,dt \\
&= \frac{e^{ia\pi}z^{-a}}{2\pi i}\int_{-i\,\infty}^{i\,\infty}\frac{\Gamma(t+a-c+1)\Gamma(t+a)}{\Gamma(a-c+1)\Gamma(a)}\Gamma(-t)z^{-t}\,dt
\end{aligned}$$

The contour passes just to the right of the imaginary axis, bending to the right to bypass the poles of $\Gamma(t+a)$ and $\Gamma(t+a-c+1)$, passing to the left of $t=0$ to avoid the poles of $\Gamma(-t)$, and then closes by joining $+i\,\infty$ to $-i\,\infty$ along the semi-circle $R\to\infty$ enclosing the left-hand side of the plane. The integral is evaluated by summing the residues at all the poles of $\Gamma(t+a)$ and $\Gamma(t+a-c+1)$. The residue of $\Gamma(-n)$ is $(-1)^n/\Gamma(n+1)$ (see page 419).

$$\begin{aligned}
U_2(a|c|z) &= \frac{e^{ia\pi}z^{-a}}{2\pi i\Gamma(a-c+1)\Gamma(a)}\int_{-i\,\infty}^{i\,\infty}\Gamma(t+a-c+1)\Gamma(t+a)\Gamma(-t)z^{-t}\,dt \\
&= \frac{e^{ia\pi}z^{-a}}{2\pi i\Gamma(a-c+1)\Gamma(a)}\left\{\sum_{n=0}^\infty \frac{(-1)^n\Gamma(c-n-1)\Gamma(1+a-c+n)}{\Gamma(n+1)}z^{1+a-c+n}\right. \\
&\qquad \left.+\sum_{n=0}^\infty \frac{\Gamma(1-c-n)(-1)^n\Gamma(a+n)}{\Gamma(n+1)}z^{a+n}\right\} \\
&= \frac{e^{ia\pi}z^{1-c}}{2\pi i\Gamma(a-c+1)\Gamma(a)}\sum_{n=0}^\infty \frac{(-1)^n\Gamma(1-[n-c+2])\Gamma(1+a-c+n)z^n}{n!} \\
&\qquad +\frac{e^{ia\pi}}{2\pi i\Gamma(a-c+1)\Gamma(a)}\sum_{n=0}^\infty \frac{(-1)^n\Gamma(1-c-n)\Gamma(a+n)z^n}{n!}
\end{aligned}$$

$$
\begin{aligned}
U_2(a|c|z) &= \frac{e^{ia\pi}z^{1-c}}{2\pi i\Gamma(a-c+1)\Gamma(a)}\sum_{n=0}^{\infty}\frac{(-1)^n\pi\Gamma(1+a-c+n)z^n}{\Gamma(2-c+n)\sin\pi(2-c+n)n!} \\
&\quad +\frac{e^{ia\pi}}{2\pi i\Gamma(a-c+1)\Gamma(a)}\sum_{n=0}^{\infty}\frac{(-1)^n\pi\Gamma(a+n)z^n}{\Gamma(c+n)\sin\pi(c+n)n!} \\
&= \frac{e^{ia\pi}z^{1-c}}{2i\Gamma(a-c+1)\Gamma(a)}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(1+a-c+n)z^n}{\Gamma(2-c+n)n!\cos n\pi\sin\pi(2-c)} \\
&\quad +\frac{e^{ia\pi}}{2i\Gamma(a-c+1)\Gamma(a)}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(a+n)z^n}{\Gamma(c+n)n!\cos n\pi\sin\pi c} \\
&= \frac{e^{ia\pi}z^{1-c}}{2i\Gamma(a-c+1)\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(1+a-c+n)z^n}{\Gamma(2-c+n)n!\sin\pi(2-c)} \\
&\quad +\frac{e^{ia\pi}}{2i\Gamma(a-c+1)\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)z^n}{\Gamma(c+n)n!\sin\pi c} \\
&= \frac{e^{ia\pi}z^{1-c}}{2i\Gamma(a-c+1)\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(1+a-c+n)\Gamma(2-c)\Gamma(c-1)z^n}{\Gamma(2-c+n)n!\pi} \\
&\quad +\frac{e^{ia\pi}}{2i\Gamma(a-c+1)\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(c)\Gamma(1-c)z^n}{\Gamma(c+n)n!\pi} \\
&= \frac{e^{ia\pi}z^{1-c}\Gamma(c-1)}{2\pi i\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(1+a-c+n)\Gamma(2-c)z^n}{\Gamma(a-c+1)\Gamma(2-c+n)n!} \\
&\quad +\frac{e^{ia\pi}\Gamma(1-c)}{2\pi i\Gamma(a-c+1)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(c)z^n}{\Gamma(a)\Gamma(c+n)n!} \\
&= \frac{e^{ia\pi}z^{1-c}\Gamma(c-1)}{2\pi i\Gamma(a)}F(1+a-c|2-c|z)+\frac{e^{ia\pi}\Gamma(1-c)}{2\pi i\Gamma(a-c+1)}F(a|c|z)
\end{aligned}
$$

Problem 6.1

$$\phi(m,n) = \frac{1}{4}[\phi(m-1,n) + \phi(m+1,n) + \phi(m,n-1) + \phi(m,n+1)]$$

which may be written as

$$0 = -\phi(m-1,n) - \phi(m+1,n) + 4\phi(m,n) - \phi(m,n-1) - \phi(m,n+1)$$

Let $\phi_i = \phi(m,n)$ where $i = 4(m-1)+n$, $1 \le m,n \le 4$, so that this last equation becomes

$$\begin{aligned} 0 &= -\phi_{i-4} - \phi_{i+4} + 4\phi_i - \phi_{i-1} - \phi_{i+1} \\ &= -\phi_{i-4} - \phi_{i-1} + 4\phi_i - \phi_{i+1} - \phi_{i+4} \end{aligned}$$

This is a system of sixteen linear equations in sixteen unknowns (the ϕ_i, $1 \le i \le 16$) where the values of $\phi(0,n)$, $\phi(5,n)$, $\phi(m,0)$ and $\phi(m,5)$ are known quantities (henceforth known as ϕ_ν, $1 \le \nu \le 16$). The system can now be written as a matrix equation

$$B_i = A_{ij}\phi_j$$

with $A_{ii} = 4$, all other A_{ij} either -1 or 0, and

$$B_i = C_{i\nu}\phi_\nu.$$

The matrix is symmetric and may be diagnolized without changing the trace, Tr $= 4^{16}$. The determinant of a diagonal matrix is equal to its trace, and in this case is not equal to zero. The system therefore has an inverse and a solution for the unknowns exists in the form

$$\begin{aligned} \phi_i &= (A^{-1})_{ij}B_j = (A^{-1})_{ij}C_{j\nu}\phi_\nu \\ &= G_{i\nu}\phi_\nu = \sum_\nu G(m,n|\nu)\phi_\nu \end{aligned}$$

The values of $G(m,n|\nu)$ represent the solution of the system when the νth boundary point has unit value ($\phi_\nu = 1$) and all other boundary points are zero.

Reflection symmetry insures that the G's for boundary points (0,4), (1,0), (1,5), (4,0), (4,5), (5,1) and (5,4) are all related to G for (0,1). Likewise, (0,3), (2,0), (2,5), (3,0), (3,5), (5,2) and (5,3) are all related to (0,2). To calculate $\phi(m,n)$ for the boundary point $\phi(0,1)$, first calculate $\phi(m,n)$ for the case $\phi(1,0) = \phi(4,0) = 1$. From symmetry, $\phi(1,n) = \phi(4,n)$, and $\phi(2,n) = \phi(3,n)$.

$$\begin{aligned} 1 &= 4\phi(1,1) - \phi(1,2) - \phi(2,1) \\ 0 &= \phi(1,1) - 4\phi(1,2) + \phi(1,3) + \phi(2,2) \\ 0 &= \phi(1,2) - 4\phi(1,3) + \phi(1,4) + \phi(2,3) \\ 0 &= \phi(1,3) - 4\phi(1,4) + \phi(2,4) \\ 0 &= \phi(1,1) - 3\phi(2,1) + \phi(2,2) \\ 0 &= \phi(1,2) + \phi(2,1) - 3\phi(2,2) + \phi(2,3) \\ 0 &= \phi(1,3) + \phi(2,2) - 3\phi(2,3) + \phi(2,4) \\ 0 &= \phi(1,4) + \phi(2,3) - 3\phi(2,4) \end{aligned}$$

This readily reduced by successive eliminations:

$$\begin{aligned} 1 &= 4\phi(1,1) - \phi(1,2) - \phi(2,1) \\ 0 &= \phi(1,1) - 4\phi(1,2) + \phi(1,3) + \phi(2,2) \\ 0 &= \phi(1,2) - 4\phi(1,3) + \phi(1,4) + \phi(2,3) \\ 0 &= 3\phi(1,3) - 11\phi(1,4) + \phi(2,3) \\ 0 &= \phi(1,1) - 3\phi(2,1) + \phi(2,2) \\ 0 &= \phi(1,2) + \phi(2,1) - 3\phi(2,2) + \phi(2,3) \\ 0 &= \frac{3}{8}\phi(1,3) + \frac{1}{8}\phi(1,4) + \frac{3}{8}\phi(2,2) - \phi(2,3) \end{aligned}$$

$$
\begin{aligned}
1 &= 4\phi(1,1) - \phi(1,2) - \phi(2,1) \\
0 &= \phi(1,1) - 4\phi(1,2) + \phi(1,3) + \phi(2,2) \\
0 &= \frac{8}{3}\phi(1,2) - \frac{29}{3}\phi(1,3) + 3\phi(1,4) + \phi(2,2) \\
0 &= 9\phi(1,3) - 29\phi(1,4) + \phi(2,2) \\
0 &= \phi(1,1) - 3\phi(2,1) + \phi(2,2) \\
0 &= \frac{8}{21}\phi(1,2) + \frac{1}{7}\phi(1,3) + \frac{1}{21}\phi(1,4) + \frac{8}{21}\phi(2,1) - \phi(2,2)
\end{aligned}
$$

$$
\begin{aligned}
1 &= 4\phi(1,1) - \phi(1,2) - \phi(2,1) \\
0 &= \frac{21}{8}\phi(1,1) - \frac{19}{2}\phi(1,2) + 3\phi(1,3) + \frac{1}{8}\phi(1,4) + \phi(2,1) \\
0 &= 8\phi(1,2) - 25\phi(1,3) + 8\phi(1,4) + \phi(2,1) \\
0 &= \phi(1,2) + 24\phi(1,3) - 76\phi(1,4) + \phi(2,1) \\
0 &= \frac{21}{55}\phi(1,1) + \frac{8}{55}\phi(1,2) + \frac{3}{55}\phi(1,3) + \frac{1}{55}\phi(1,4) - \phi(2,1)
\end{aligned}
$$

$$
\begin{aligned}
55 &= 199\phi(1,1) - 63\phi(1,2) - 3\phi(1,3) - \phi(1,4) \\
0 &= 21\phi(1,1) - \frac{588}{9}\phi(1,2) + \frac{192}{9}\phi(1,3) + \phi(1,4) \\
0 &= \frac{7}{1347}\phi(1,1) + \frac{448}{4041}\phi(1,2) - \frac{458}{1347}\phi(1,3) + \phi(1,4) \\
0 &= \frac{1}{199}\phi(1,1) + \frac{9}{597}\phi(1,2) + \frac{431}{1393}\phi(1,3) - \phi(1,4)
\end{aligned}
$$

Switch to decimals.

$$
\begin{aligned}
55 &= 199\phi(1,1) - 63\phi(1,2) - 3\phi(1,3) - \phi(1,4) \\
0 &= 21\phi(1,1) - 65.33333333\phi(1,2) + 21.33333333\phi(1,3) + \phi(1,4) \\
0 &= 0.005196733\phi(1,1) + 0.110863648\phi(1,2) - 0.340014848\phi(1,3) + \phi(1,4) \\
0 &= 0.005025126\phi(1,1) + 0.015075377\phi(1,2) + 0.309404164\phi(1,3) - \phi(1,4)
\end{aligned}
$$

$$
\begin{aligned}
16.61930586 &= 60.13015184\phi(1,1) - 19.04121475\phi(1,2) - \phi(1,3) \\
0 &= 0.864071004\phi(1,1) - 3.125741961\phi(1,2) + \phi(1,3) \\
0 &= 0.00076615\phi(1,1) + 0.339872845\phi(1,2) - \phi(1,3)
\end{aligned}
$$

$$
\begin{aligned}
0.749733311 &= 2.751583072\phi(1,1) - \phi(1,2) \\
0 &= 0.312912247\phi(1,1) - \phi(1,2)
\end{aligned}
$$

$$
\begin{aligned}
0.30123508 &= \phi(1,1) = \phi(4,1) \\
0.095342534 &= \phi(1,2) = \phi(4,2) \\
0.032639641 &= \phi(1,3) = \phi(4,3) \\
0.0130795 &= \phi(1,4) = \phi(4,4) \\
0.13351497 &= \phi(2,1) = \phi(3,1) \\
0.092330984 &= \phi(2,2) = \phi(3,2) \\
0.048498922 &= \phi(2,3) = \phi(3,3) \\
0.020526141 &= \phi(2,4) = \phi(3,4)
\end{aligned}
$$

Next, calculate $\phi(m,n)$ for the case $\phi(1,0) = -\phi(4,0) = 1$. From symmetry, $\phi(1,n) = -\phi(4,m)$, $\phi(2,n) = -\phi(3,n)$ and

$$\begin{aligned}
0 &= 4\phi(1,1) - \phi(1,2) - \phi(2,1)\\
0 &= \phi(1,1) - 4\phi(1,2) + \phi(1,3) + \phi(2,2)\\
0 &= \phi(1,2) - 4\phi(1,3) + \phi(1,4) + \phi(2,3)\\
0 &= \phi(1,3) - 4\phi(1,4) + \phi(2,4)\\
1 &= \phi(1,1) - 5\phi(2,1) + \phi(2,2)\\
0 &= \phi(1,2) + \phi(2,1) - 5\phi(2,2) + \phi(2,3)\\
0 &= \phi(1,3) + \phi(2,2) - 5\phi(2,3) + \phi(2,4)\\
0 &= \phi(1,4) + \phi(2,3) - 5\phi(2,4)
\end{aligned}$$

This readily reduced by successive eliminations:

$$\begin{aligned}
0 &= 4\phi(1,1) - \phi(1,2) - \phi(2,1)\\
0 &= \phi(1,1) - 4\phi(1,2) + \phi(1,3) + \phi(2,2)\\
0 &= \phi(1,2) - 4\phi(1,3) + \phi(1,4) + \phi(2,3)\\
0 &= 5\phi(1,3) - 9\phi(1,4) + \phi(2,3)\\
1 &= \phi(1,1) - 5\phi(2,1) + \phi(2,2)\\
0 &= \phi(1,2) + \phi(2,1) - 3\phi(2,2) + \phi(2,3)\\
0 &= \frac{5}{24}\phi(1,3) + \frac{1}{24}\phi(1,4) + \frac{5}{24}\phi(2,2) - \phi(2,3)
\end{aligned}$$

$$\begin{aligned}
0 &= 4\phi(1,1) - \phi(1,2) - \phi(2,1)\\
0 &= \phi(1,1) - 4\phi(1,2) + \phi(1,3) + \phi(2,2)\\
0 &= \frac{24}{5}\phi(1,2) - \frac{91}{5}\phi(1,3) + 5\phi(1,4) + \phi(2,2)\\
0 &= 25\phi(1,3) - 93\phi(1,4) + \phi(2,2)\\
1 &= \phi(1,1) - 5\phi(2,1) + \phi(2,2)\\
0 &= \frac{24}{67}\phi(1,2) + \frac{5}{67}\phi(1,3) + \frac{1}{67}\phi(1,4) + \frac{24}{67}\phi(2,1) - \phi(2,2)
\end{aligned}$$

$$\begin{aligned}
0 &= 4\phi(1,1) - \phi(1,2) - \phi(2,1)\\
0 &= \frac{67}{24}\phi(1,1) - \frac{41}{16}\phi(1,2) + 3\phi(1,3) + \frac{1}{24}\phi(1,4) + \phi(2,1)\\
0 &= \frac{72}{5}\phi(1,2) - \frac{253}{5}\phi(1,3) + 14\phi(1,4) + \phi(2,1)\\
0 &= \phi(1,2) + 70\phi(1,3) - \frac{3115}{12}\phi(1,4) + \phi(2,1)\\
\frac{67}{311} &= \frac{24}{311}\phi(1,1) + \frac{24}{311}\phi(1,2) + \frac{15}{311}\phi(1,3) + \frac{1}{311}\phi(1,4) - \phi(2,1)
\end{aligned}$$

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0 &= \frac{1}{199}\phi(1,1) + \frac{9}{597}\phi(1,2) + \frac{431}{1393}\phi(1,3) - \phi(1,4)
\end{aligned}$$

Switch to decimals.

$$\begin{aligned}
55 &= 199\phi(1,1) - 63\phi(1,2) - 3\phi(1,3) - \phi(1,4)\\
0 &= 21\phi(1,1) - 65.33333333\phi(1,2) + 21.33333333\phi(1,3) + \phi(1,4)\\
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0 &= 0.005025126\phi(1,1) + 0.015075377\phi(1,2) + 0.309404164\phi(1,3) - \phi(1,4)
\end{aligned}$$

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$$0 = 0.00076615\phi(1,1) + 0.339872845\phi(1,2) - \phi(1,3)$$

$$0.749733311 = 2.751583072\phi(1,1) - \phi(1,2)$$
$$0 = 0.312912247\phi(1,1) - \phi(1,2)$$

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