Problem Solutions for The Quantum Theory of Fields, Vol. I

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August 29, 2005

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Chapter II, Problem 1.

$$\mathbf{p} = \begin{pmatrix} 0 \\ p \\ 0 \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{\gamma^2 - 1} \\ 0 \\ m\gamma \end{pmatrix} \qquad E^2 - p^2 = m^2 \qquad \gamma = \sqrt{\frac{p^2 + m^2}{m^2}}$$

$$L(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & \sqrt{\gamma^2 - 1} \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{\gamma^2 - 1} & 0 & \gamma \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \bar{\gamma} & \sqrt{\bar{\gamma}^2 - 1} \\ 0 & 0 & \sqrt{\bar{\gamma}^2 - 1} & \bar{\gamma} \end{pmatrix}$$

$$\bar{\gamma} = (1 - \mathbf{v} \cdot \mathbf{v})^{-1/2} \quad (c = 1)$$

$$\Lambda p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \bar{\gamma} & \sqrt{\bar{\gamma}^2 - 1} \\ 0 & 0 & \sqrt{\bar{\gamma}^2 - 1} & \bar{\gamma} \end{pmatrix} \begin{pmatrix} 0 \\ m\sqrt{\gamma^2 - 1} \\ 0 \\ m\gamma \end{pmatrix} = \begin{pmatrix} 0 \\ m\sqrt{\gamma^2 - 1} \\ 0 \\ m\gamma\sqrt{\bar{\gamma}^2 - 1} \\ m\gamma\bar{\gamma} \end{pmatrix}$$

$$\begin{aligned} \widehat{(\Lambda p)}_x &= 0 \\ \widehat{(\Lambda p)}_y &= \frac{m\sqrt{\gamma^2 - 1}}{\sqrt{(m^2(\gamma^2 - 1) + m^2\gamma^2(\bar{\gamma}^2 - 1)}} = \frac{\sqrt{\gamma^2 - 1}}{\sqrt{\gamma^2\bar{\gamma}^2 - 1}} \\ \widehat{(\Lambda p)}_z &= \frac{\gamma\sqrt{\bar{\gamma}^2 - 1}}{\sqrt{\gamma^2\bar{\gamma}^2 - 1}} \end{aligned}$$

$$L^{-1}(\Lambda p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 + (\gamma\bar{\gamma} - 1)\frac{\gamma^2 - 1}{\gamma^2\bar{\gamma}^2 - 1} & (\gamma\bar{\gamma} - 1)\frac{\gamma\sqrt{(\gamma^2 - 1)(\bar{\gamma}^2 - 1)}}{\gamma^2\bar{\gamma}^2 - 1} & -\sqrt{\gamma^2 - 1} \\ 0 & (\gamma\bar{\gamma} - 1)\frac{\gamma\sqrt{(\gamma^2 - 1)(\bar{\gamma}^2 - 1)}}{\gamma^2\bar{\gamma}^2 - 1} & 1 + (\gamma\bar{\gamma} - 1)\frac{\gamma^2 - 1}{\gamma^2\bar{\gamma}^2 - 1} & -\gamma\sqrt{\bar{\gamma}^2 - 1} \\ 0 & -\sqrt{\gamma^2 - 1} & -\gamma\sqrt{\bar{\gamma}^2 - 1} & \gamma\bar{\gamma} \end{pmatrix}$$

$$\Lambda L(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & \sqrt{\gamma^2 - 1} \\ 0 & \sqrt{(\gamma^2 - 1)(\bar{\gamma}^2 - 1)} & \bar{\gamma} & \gamma\sqrt{\bar{\gamma}^2 - 1} \\ 0 & \bar{\gamma}\sqrt{\gamma^2 - 1} & \sqrt{\bar{\gamma}^2 - 1} & \gamma\bar{\gamma} \end{pmatrix}$$

$$L^{-1}(\Lambda p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\gamma(\gamma+\bar{\gamma})}{\gamma\bar{\gamma}+1} & \frac{\gamma\sqrt{(\gamma^2-1)(\bar{\gamma}^2-1)}}{\gamma\bar{\gamma}+1} & -\sqrt{\gamma^2-1} \\ 0 & \frac{\gamma\sqrt{(\gamma^2-1)(\bar{\gamma}^2-1)}}{\gamma\bar{\gamma}+1} & 1+\frac{\gamma^2(\bar{\gamma}^2-1)}{\gamma\bar{\gamma}+1} & -\gamma\sqrt{\bar{\gamma}^2-1} \\ 0 & -\sqrt{\gamma^2-1} & -\gamma\sqrt{\bar{\gamma}^2-1} & \gamma\bar{\gamma} \end{pmatrix}$$

$$L^{-1}(\Lambda p)\Lambda L(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\gamma+\bar{\gamma}}{\gamma\bar{\gamma}+1} & -\frac{\sqrt{(\gamma^2-1)(\bar{\gamma}^2-1)}}{\gamma\bar{\gamma}+1} & 0 \\ 0 & \frac{\sqrt{(\gamma^2-1)(\bar{\gamma}^2-1)}}{\gamma\bar{\gamma}+1} & \frac{\gamma+\bar{\gamma}}{\gamma\bar{\gamma}+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is a rotation about the x axis by an angle $\theta = \cos^{-1}\left(\frac{\gamma+\bar{\gamma}}{\gamma\bar{\gamma}+1}\right)$ with the new coordinate system rotated clockwise with respect to the old (Thomas precession).

$$\begin{aligned} W(\Lambda,p) &= e^{i\theta J_{23}} & J_-^{(j)} &= J_{23}^{(j)} - iJ_{31}^{(j)} \\ D_{\sigma,\sigma'}^{(j)} &= e^{i\theta(J_{23}^{(j)})_{\sigma'\sigma}} & J_+^{(j)} &= J_{23}^{(j)} + iJ_{31}^{(j)} \\ \left(J_{23}^{(j)}\right)_{\sigma'\sigma} &= \tfrac{1}{2}\left(J_+^{(j)} + J_-^{(j)}\right)_{\sigma'\sigma} & J_+^{(j)}\psi_{k,\sigma} &= \sqrt{(j-\sigma)(j+\sigma+1)}\psi_{k,\sigma+1} \\ D_{\sigma'\sigma}^{(j)} &= e^{\frac{i\theta}{2}(J_+^{(j)}+J_-^{(j)})_{\sigma'\sigma}} & J_-^{(j)}\psi_{k,\sigma} &= \sqrt{(j+\sigma)(j-\sigma+1)}\psi_{k,\sigma-1} \end{aligned}$$

$$D_{\sigma'\sigma}^{(j)} = e^{\frac{i\theta}{2}\left(\delta_{\sigma',\sigma+1}\sqrt{(j-\sigma)(j+\sigma+1)}+\delta_{\sigma',\sigma-1}\sqrt{(j+\sigma)(j-\sigma+1)}\right)}$$

For $j = 1$:

$$\begin{aligned} \sigma = 1: &\quad \sqrt{(j-\sigma)(j+\sigma+1)} = 0 \quad \sqrt{(j+\sigma)(j-\sigma+1)} = \sqrt{2} \\ \sigma = 0: &\quad \sqrt{(j-\sigma)(j+\sigma+1)} = \sqrt{2} \quad \sqrt{(j+\sigma)(j-\sigma+1)} = \sqrt{2} \\ \sigma = -1: &\quad \sqrt{(j-\sigma)(j+\sigma+1)} = \sqrt{2} \quad \sqrt{(j+\sigma)(j-\sigma+1)} = 0 \end{aligned}$$

$$\begin{aligned} \delta_{\sigma',\sigma+1}\sqrt{(j-\sigma)(j+\sigma+1)} &= \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \\ \delta_{\sigma',\sigma-1}\sqrt{(j+\sigma)(j-\sigma+1)} &= \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \end{aligned}$$

$$D_{\sigma'\sigma}^{(1)} = \exp\left\{\frac{i\theta}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right\} = \sum_{n=0}^{\infty}\left(\frac{i\theta}{2}\right)^n \frac{1}{n!}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^n$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\begin{aligned}
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^4 &= \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}^2 \\
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^5 &= 2\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= 2\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} = 4\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{2n} &= 2^{n-1}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{2(n+1)} &= 2^{n-1}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 \\
&= 2^{n-1}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \\
&= 2^{n}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{except } n = 0. \\
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{2n+1} &= 2^{n-1}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= 2^{n}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
D^{(1)}_{\sigma'\sigma} &= \exp\left\{\frac{i\theta}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right\} = \sum_{n=0}^{\infty}\left(\frac{i\theta}{\sqrt{2}}\right)^{2n}\frac{1}{(2n)!}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{2n} \\
&\qquad + \sum_{n=0}^{\infty}\left(\frac{i\theta}{\sqrt{2}}\right)^{2n+1}\frac{1}{(2n+1)!}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{2n+1} \\
&= \sum_{n=0}^{\infty}\left(\frac{i\theta}{\sqrt{2}}\right)^{2n}\frac{2^{n-1}}{(2n)!}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \sum_{n=0}^{\infty}\left(\frac{i\theta}{\sqrt{2}}\right)^{2n+1}\frac{2^{n}}{(2n+1)!}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sum_{n=1}^{\infty}\left(\frac{i\theta}{\sqrt{2}}\right)^{2n}\frac{2^{n-1}}{(2n)!}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
&\qquad + \sum_{n=0}^{\infty}\left(\frac{i\theta}{\sqrt{2}}\right)^{2n+1}\frac{2^{n}}{(2n+1)!}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}(\cos\theta - 1) + \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
U(\Lambda)\psi_{p,\sigma} &= \sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma'=-1}^{1} D^{(1)}_{\sigma'\sigma}\psi_{\Lambda p,\sigma'} \\
U(\Lambda)\psi_{p,+1} &= \sqrt{\bar{\gamma}}\left\{\frac{\cos\theta - 1}{2}\psi_{\Lambda p,-1} + \frac{i}{\sqrt{2}}\sin\theta\psi_{\Lambda p,0} + \frac{\cos\theta + 1}{2}\psi_{\Lambda p,+1}\right\} \\
&= \sqrt{\bar{\gamma}}\left\{\frac{\frac{\gamma+\bar{\gamma}}{\gamma\bar{\gamma}+1} - 1}{2}\psi_{\Lambda p,-1} + \frac{i}{\sqrt{2}}\frac{\sqrt{\gamma^2\bar{\gamma}^2 - \gamma^2 - \bar{\gamma}^2 + 1}}{\gamma\bar{\gamma}+1}\psi_{\Lambda p,0} + \frac{\frac{\gamma+\bar{\gamma}}{\gamma\bar{\gamma}+1} + 1}{2}\psi_{\Lambda p,+1}\right\} \\
U(\Lambda)\psi_{p,0} &= \sqrt{\bar{\gamma}}\left\{\frac{i}{\sqrt{2}}\frac{\sqrt{\gamma^2\bar{\gamma}^2 - \gamma^2 - \bar{\gamma}^2 + 1}}{\gamma\bar{\gamma}+1}\psi_{\Lambda p,+1} + \frac{\gamma+\bar{\gamma}}{\gamma\bar{\gamma}}\psi_{\Lambda p,0}\right. \\
&\qquad \left. + \frac{i}{\sqrt{2}}\frac{\sqrt{\gamma^2\bar{\gamma}^2 - \gamma^2 - \bar{\gamma}^2 + 1}}{\gamma\bar{\gamma}+1}\psi_{\Lambda p,-1}\right\} \\
U(\Lambda)\psi_{p,-1} &= \sqrt{\bar{\gamma}}\left\{\frac{\frac{\gamma+\bar{\gamma}}{\gamma\bar{\gamma}+1} - 1}{2}\psi_{\Lambda p,+1} + \frac{i}{\sqrt{2}}\frac{\sqrt{\gamma^2\bar{\gamma}^2 - \gamma^2 - \bar{\gamma}^2 + 1}}{\gamma\bar{\gamma}+1}\psi_{\Lambda p,0} + \frac{\frac{\gamma+\bar{\gamma}}{\gamma\bar{\gamma}+1} + 1}{2}\psi_{\Lambda p,_1}\right\}
\end{aligned}
$$

The particle will appear to be a superposition of all allowed values of σ for a spin-1 boson of mass m.

Chapter II, Problem 2.

$L(p)$ is a boost in the z direction followed by a rotation which brings the momentum 3-vector to the desired direction:

$$\mathbf{p} = \begin{pmatrix} p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix}$$

$$\begin{aligned} R(\theta,\phi) &= \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi\cos\theta & -\sin\phi & \cos\phi\sin\theta & 0 \\ \sin\phi\cos\theta & \cos\phi & \sin\phi\sin\theta & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

The z boost is

$$\begin{aligned} B(p) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\beta & \sinh\beta \\ 0 & 0 & \sinh\beta & \cosh\beta \end{pmatrix} \\ L(p) &= \begin{pmatrix} \cos\phi\cos\theta & -\sin\phi & \cos\phi\sin\theta & 0 \\ \sin\phi\cos\theta & \cos\phi & \sin\phi\sin\theta & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\beta & \sinh\beta \\ 0 & 0 & \sinh\beta & \cosh\beta \end{pmatrix} \\ L(p) &= \begin{pmatrix} \cos\phi\cos\theta & -\sin\phi & \cos\phi\sin\theta\cosh\beta & \cos\phi\sin\theta\sinh\beta \\ \sin\phi\cos\theta & \cos\phi & \sin\phi\sin\theta\cosh\beta & \sin\phi\sin\theta\sinh\beta \\ -\sin\theta & 0 & \cos\theta\cosh\beta & \cos\theta\sinh\beta \\ 0 & 0 & \sinh\beta & \cosh\beta \end{pmatrix} \\ \mathbf{p} &= L(p)\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\phi\sin\theta(\cosh\beta+\sinh\beta) \\ \sin\phi\sin\theta(\cosh\beta+\sinh\beta) \\ \cos\theta(\cosh\beta+\sinh\beta) \\ \cosh\beta+\sinh\beta \end{pmatrix} \end{aligned}$$

For $\mathbf{p}$ in the y direction,

$$\cos\theta = \cos\phi = 0,$$

$$0 \le \theta \le \pi \to \theta = \frac{\pi}{2}, \quad \sin\theta = 1$$

and

$$\sin\phi\sin\theta > 0 \to \sin\phi > 0 \to \phi = \frac{\pi}{2}.$$

$$\begin{aligned}
L(p) &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & \cosh\beta & \sinh\beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sinh\beta & \cosh\beta \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\bar{\beta} & -\sinh\bar{\beta} \\ 0 & 0 & -\sinh\bar{\beta} & \cosh\bar{\beta} \end{pmatrix} \\
\Lambda p &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\bar{\beta} & -\sinh\bar{\beta} \\ 0 & 0 & -\sinh\bar{\beta} & \cosh\bar{\beta} \end{pmatrix} \begin{pmatrix} 0 \\ p \\ 0 \\ p \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ p \\ -p\sinh\bar{\beta} \\ p\cosh\bar{\beta} \end{pmatrix} = \begin{pmatrix} \cos\bar{\phi}\sin\bar{\theta}(\cosh\bar{\bar{\beta}} + \sinh\bar{\bar{\beta}}) \\ \sin\bar{\phi}\cos\bar{\theta}(\cosh\bar{\bar{\beta}} + \sinh\bar{\bar{\beta}}) \\ \cos\bar{\theta}(\cosh\bar{\bar{\beta}} + \sinh\bar{\bar{\beta}}) \\ \cosh\bar{\bar{\beta}} + \sinh\bar{\bar{\beta}} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\cos\bar{\theta} = -\tanh\bar{\beta} \le 0 \quad &\to \quad \frac{\pi}{2} \le \bar{\theta} \le \pi \to \sin\bar{\theta} \ge 0 \\
p > 0 \quad &\to \quad \sin\bar{\phi}\sin\bar{\theta} > 0 \to \sin\bar{\phi} > 0 \to 0 \le \bar{\phi} \le \pi \\
\cos\bar{\phi} = 0 \quad &\to \quad \bar{\phi} = \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
p\cosh\bar{\beta} &= \cosh\bar{\bar{\beta}} + \sinh\bar{\bar{\beta}} \\
\sinh\bar{\bar{\beta}} &= p\cosh\bar{\beta} - \cosh\bar{\bar{\beta}} \\
\sinh^2\bar{\bar{\beta}} &= p^2\cosh^2\bar{\beta} - 2p\cosh\bar{\beta}\cosh\bar{\bar{\beta}} + \cosh^2\bar{\bar{\beta}} \\
p^2\cosh^2\bar{\beta} + 1 &= 2p\cosh\bar{\beta}\cosh\bar{\bar{\beta}} \\
\cosh\bar{\bar{\beta}} &= \frac{p^2\cosh^2\bar{\beta} + 1}{2p\cosh\bar{\beta}} \\
\sinh\bar{\beta} &= p\cosh\bar{\beta} - \cosh\bar{\bar{\beta}} = p\cosh\bar{\beta} - \frac{p^2\cosh^2\bar{\beta} + 1}{2p\cosh\bar{\beta}} \\
&= \frac{p^2\cosh^2\bar{\beta} - 1}{2p\cosh\bar{\beta}}
\end{aligned}$$

$$
\begin{aligned}
L(\Lambda p) &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -\tanh\bar\beta & 0 & \frac{p^2\cosh^2\bar\beta+1}{2p\cosh^2\bar\beta} & \frac{p^2\cosh^2\bar\beta-1}{2p\cosh^2\bar\beta} \\ -\frac{1}{\cosh\bar\beta} & 0 & \frac{-\sinh\bar\beta(p^2\cosh^2\bar\beta+1)}{2p\cosh^2\bar\beta} & \frac{-\sinh\bar\beta(p^2\cosh^2\bar\beta-1)}{2p\cosh^2\bar\beta} \\ & 0 & \frac{p^2\cosh^2\bar\beta-1}{2p\cosh^2\bar\beta} & \frac{p^2\cosh^2\bar\beta+1}{2p\cosh^2\bar\beta} \end{pmatrix} \\
L^{-1}(\Lambda p) &= \begin{pmatrix} 0 & -\tanh\bar\beta & -\frac{1}{\cosh\bar\beta} & 0 \\ -1 & 0 & 0 & 0 \\ 0 & \frac{p^2\cosh^2\bar\beta+1}{2p\cosh^2\bar\beta} & -\sinh\bar\beta\frac{p^2\cosh^2\bar\beta+1}{2p\cosh^2\bar\beta} & -\frac{p^2\cosh^2\bar\beta-1}{2p\cosh^2\bar\beta} \\ 0 & -\frac{p^2\cosh^2\bar\beta-1}{2p\cosh^2\bar\beta} & \sinh\bar\beta\frac{p^2\cosh^2\bar\beta-1}{2p\cosh^2\bar\beta} & \frac{p^2\cosh^2\bar\beta+1}{2p\cosh^2\bar\beta} \end{pmatrix} \\
\Lambda L(p) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\bar\beta & -\sinh\bar\beta \\ 0 & 0 & -\sinh\bar\beta & \cosh\bar\beta \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & \cosh\beta & \sinh\beta \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \sinh\beta & \cosh\beta \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & \cosh\beta & \sinh\beta \\ -\cosh\bar\beta & 0 & -\sinh\beta\sinh\bar\beta & -\cosh\beta\sinh\bar\beta \\ \sinh\bar\beta & 0 & \sinh\beta\cosh\bar\beta & \cosh\beta\cosh\bar\beta \end{pmatrix}
\end{aligned}
$$

$$
L^{-1}\left(\Lambda p\right)\left(L(p)\right) =
$$

$$
\begin{pmatrix} 1 & 0 & -\tanh\bar\beta(\cosh\beta-\sinh\beta) & \tanh\beta(\sinh\beta-\cosh\beta) \\ 0 & 1 & 0 & 0 \\ \frac{\tanh\bar\beta}{\cosh\beta+\sinh\beta} & 0 & \frac{\cosh\beta-\sinh\beta}{2(\cosh\beta+\sinh\beta)\cosh^2\bar\beta}+\frac{1}{2}+\frac{\sinh\beta}{\cosh\beta+\sinh\beta} & \frac{\sinh\beta-\cosh\beta}{2(\cosh\beta+\sinh\beta)\cosh^2\bar\beta}-\frac{1}{2}+\frac{\cosh\beta}{\cosh\beta+\sinh\beta} \\ \frac{\tanh\bar\beta}{\cosh\beta+\sinh\beta} & 0 & \frac{\cosh\beta-\sinh\beta}{2(\cosh\beta+\sinh\beta)\cosh^2\bar\beta}+\frac{1}{2}-\frac{\cosh\beta}{\cosh\beta+\sinh\beta} & \frac{\sinh\beta-\cosh\beta}{2(\cosh\beta+\sinh\beta)\cosh^2\bar\beta}+\frac{1}{2}-\frac{\cosh\beta}{\cosh\beta+\sinh\beta} \end{pmatrix}
$$

$$
\frac{1}{p} = \frac{1}{\cosh\beta+\sinh\beta} = \frac{\cosh\beta-\sinh\beta}{\cosh^2\beta-\sinh^2\beta} = \cosh\beta-\sinh\beta
$$

$$
L^{-1}\left(\Lambda p\right)\left(L(p)\right) = \begin{pmatrix} 1 & 0 & -\frac{\tanh\bar\beta}{p} & \frac{\tanh\bar\beta}{p} \\ 0 & 1 & 0 & 0 \\ \frac{\tanh\bar\beta}{p} & 0 & \frac{1}{2}+\frac{p^2-1}{2p^2}+\frac{1}{2p^2\cosh^2\bar\beta} & -\frac{1}{2}+\frac{p^2+1}{2p^2}-\frac{1}{2p^2\cosh^2\bar\beta} \\ \frac{\tanh\bar\beta}{p} & 0 & \frac{1}{2}-\frac{p^2+1}{2p^2}+\frac{1}{2p^2\cosh^2\bar\beta} & \frac{1}{2}-\frac{p^2+1}{2p^2}-\frac{1}{2p^2\cosh^2\bar\beta} \end{pmatrix}
$$

$$
\begin{aligned}
p = \cosh\beta+\sinh\beta \quad \cosh\beta &= \frac{p+\frac{1}{p}}{2} = \frac{p^2+1}{2p} \\
\frac{1}{p} = \cosh\beta-\sinh\beta \quad \sinh\beta &= \frac{p-\frac{1}{p}}{2} = \frac{p^2-1}{2p}
\end{aligned}
$$

$$L^{-1}\left(\Lambda p\right)\left(L(p)\right) = \begin{pmatrix} 1 & 0 & -\frac{\tanh\bar{\beta}}{p} & \frac{\tanh\bar{\beta}}{p} \\ 0 & 1 & 0 & 0 \\ \frac{\tanh\bar{\beta}}{p} & 0 & 1-\dfrac{\tanh^2\bar{\beta}}{2p^2} & \dfrac{\tanh^2\bar{\beta}}{p} \\ \frac{\tanh\bar{\beta}}{p} & 0 & -\dfrac{\tanh^2\bar{\beta}}{2p^2} & 1+\dfrac{\tanh^2\bar{\beta}}{2p^2} \end{pmatrix}$$

$$\alpha = \frac{\tanh\bar{\beta}}{p} \quad \beta = 0 \quad \theta = 0$$

$$\begin{aligned} U(\Lambda)\Psi^{p,\sigma} &= \sqrt{\frac{(\Lambda p)^0}{p^0}} e^{i\sigma\theta(\Lambda,p)}\Psi_{\Lambda p,\sigma} \\ &= \sqrt{\frac{p\cosh\bar{\beta}}{p}}\Psi^{\Lambda p,\sigma} = \sqrt{\cosh\bar{\beta}}\Psi^{\Lambda p,\sigma} \end{aligned}$$

$\Psi_p = \Psi_{p,+1} + \Psi_{p,-1}$ since Ψ_p is linearly polarized. The phase factor is measured from the y-z plane and is $\theta = 0$.

$$\begin{aligned} U(\Lambda)\Psi^p &= U(\Lambda)\left\{\Psi_{p,1} + \Psi_{p,-1}\right\} \\ &= U(\Lambda)\Psi_{p,1} + U(\Lambda)\Psi_{p,-1} \\ &= \sqrt{\cosh\bar{\beta}}\left\{e^{i(+1)\theta}\Psi_{\Lambda p,+1} + e^{i(-1)\theta}\Psi_{\Lambda p,-1}\right\} \\ &= \sqrt{\cosh\bar{\beta}}\left\{\Psi_{\Lambda p,+1} + \Psi_{\Lambda p,-1}\right\} \\ &= \sqrt{\cosh\bar{\beta}}\Psi_{\Lambda p} \end{aligned}$$

$\mathcal{O}'$ sees a photon of momentum $\mathbf{p}=\begin{pmatrix} 0 \\ p \\ -p\sinh\bar{\beta} \\ p\cosh\bar{\beta} \end{pmatrix}$, linearly polarized in the y'-z' plane.

Chapter II, Problem 3.

The most general Galilean transformation is a 3-rotation combined with a boost $\mathbf{v}t$, followed by a displacement $\mathbf{a}$:

$$
\begin{aligned}
\mathbf{x} &= \hat{\theta}\mathbf{x} + \mathbf{v}t + \mathbf{a} \qquad t' = t + \tau \\
\begin{pmatrix} \mathbf{x}' \\ t' \end{pmatrix} &= \begin{pmatrix} \hat{\theta} & \mathbf{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} + \begin{pmatrix} \mathbf{a} \\ \tau \end{pmatrix} \\
\begin{pmatrix} \mathbf{x}'' \\ t'' \end{pmatrix} &= \begin{pmatrix} \hat{\theta}_2 & \mathbf{v}_2 \\ 0 & 1 \end{pmatrix} \left\{ \begin{pmatrix} \hat{\theta}_1 & \mathbf{v}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} + \begin{pmatrix} \mathbf{a}_1 \\ \tau_1 \end{pmatrix} \right\} + \begin{pmatrix} \mathbf{a}_2 \\ \tau_2 \end{pmatrix} \\
&= \begin{pmatrix} \hat{\theta}_2 & \mathbf{v}_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\theta}_1 & \mathbf{v}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} + \begin{pmatrix} \hat{\theta}_2 & \mathbf{v}_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \tau_1 \end{pmatrix} + \begin{pmatrix} \mathbf{a}_2 \\ \tau_2 \end{pmatrix} \\
G(\theta_2, a_2)G(\theta_1, a_1) &= G(\theta_2\theta_1, \theta_2\theta_1 + a_2); \quad \theta = \begin{pmatrix} \hat{\theta} & \mathbf{v} \\ 0 & 1 \end{pmatrix} \\
G^{-1}(\theta, a) &= G(\theta^{-1}, -\theta^{-1}a); \quad a = \begin{pmatrix} \mathbf{a} \\ \tau \end{pmatrix} \\
\theta_2\theta_1 &= \begin{pmatrix} \hat{\theta}_2 & \mathbf{v}_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\theta}_1 & \mathbf{v}_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \hat{\theta}_2\hat{\theta}_1 & \hat{\theta}_2\mathbf{v}_1 + \mathbf{v}_2 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

If $\theta_1 = \theta^{-1}$ and $\theta_2 = \theta$, then $\hat{\theta}\hat{\theta}^{-1} = 1$ and

$$
\begin{aligned}
\hat{\theta}\mathbf{v}_1 + \mathbf{v}_2 &= 0 \\
\hat{\theta}\mathbf{v}_1 &= -\mathbf{v}_2 \\
\mathbf{v}_1 &= -\hat{\theta}^{-1}\mathbf{v}_2 \\
\theta^{-1} &= \begin{pmatrix} \hat{\theta}^{-1} & -\hat{\theta}^{-1}\mathbf{v} \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

Since $\hat{\theta}$ is a 3-rotation, $\hat{\theta}^{-1} = \hat{\theta}^T$.

$$
\begin{aligned}
G(\theta, a)G(1+\omega, \epsilon)G(\theta^{-1}, -\theta^{-1}a) &= G(\theta, a)G((1+\omega)\theta^{-1}, (1+\omega)(-\theta^{-1}a) + \epsilon) \\
&= g(\theta(1+\omega)\theta^{-1}, \theta\{(1+\omega)\theta^{-1}(-a) + \epsilon\} + a) \\
&= g(\theta\theta^{-1} + \theta\omega\theta^{-1}, -\theta(1+\omega)\theta^{-1}a + \theta\epsilon + a) \\
&= G(1 + \theta\omega\theta^{-1}, -\theta\omega\theta^{-1}a + \theta\epsilon)
\end{aligned}
$$

For transformations near the identity,

$$
\begin{aligned}
\theta &= \begin{pmatrix} \hat{\theta}_{ij} & v_i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \delta_{ij} + \omega_{ij} & \omega_{i0} \\ 0 & 1 \end{pmatrix} \\
\theta^{-1} &= \begin{pmatrix} \hat{\theta}^T_{jk} & -\hat{\theta}^T_{jl} v_l \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \hat{\theta}_{kj} & -\hat{\theta}_{jl} v_l \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \delta_{kj} + \omega_{kj} & -(\delta_{lj} + \omega_{lj})(\omega_{l0}) \\ 0 & 1 \end{pmatrix} \\
\theta\theta^{-1} = \delta_{\alpha\beta} &= \begin{pmatrix} (\delta_{ij} + \omega_{ij})(\delta_{kj} + \omega_{kj}) & -(\delta_{ij} + \omega_{ij})(\delta_{lj} + \omega_{lj})(\omega_{l0}) + \omega_{i0} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \delta_{ij} + \omega_{ik} + \omega_{ki} & -(\delta_{il} + \omega_{il} + \omega_{li})\omega_{l0} + \omega_{i0} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \delta_{ik} + \omega_{ik} + \omega_{ki} & -\omega_{i0} - (\omega_{il}\omega_{l0} + \omega_{li}\omega_{l0}) + \omega_{i0} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \delta_{ik} + \omega_{ik} + \omega_{ki} & -\omega_{l0}(\omega_{il} + \omega_{li}) \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

This implies $\omega_{ik} = -\omega_{ki}$ with $\omega_{0l} = 0$ and $\omega_{00} = 0$.

$$
\begin{aligned}
G(1+\omega,\epsilon) &= 1 + \frac{1}{2} i\omega_{ij} J^{ij} + \frac{1}{2} i\omega_{i0} J^{i0} - i\epsilon_\alpha P^\alpha \\
G(\theta,a)G(1+\omega,\epsilon)G(\theta^{-1},-\theta^{-1}a) &= G(\theta,a)G((1+\omega)\theta^{-1},(1+\omega)(-\theta^{-1}a)+\epsilon) \\
&= G(\theta(1+\omega)\theta^{-1},\theta\{(1+\omega)\theta^{-1}(-a)+\epsilon\}+1) \\
&= G(\theta\theta^{-1}+\theta\omega\theta^{-1},-\theta(1+\omega)\theta^{-1}a+\theta\epsilon+a) \\
&= G(1+\theta\omega\theta^{-1},-\theta\omega\theta^{-1}a+\theta\epsilon)
\end{aligned}
$$

$$
G(\theta,a)\left\{1 + \frac{1}{2} i\omega_{ij} J^{ij} + \frac{1}{2} i\omega_{i0} J^{i0} - i\epsilon_\alpha P^\alpha\right\} G(\theta^{-1},\theta^{-1}a) =
$$

$$
1 + \frac{1}{2} i(\theta\omega\theta^{-1})_{\alpha\beta} J^{\alpha\beta} - i(-\theta\omega\theta^{-1}a + \theta\epsilon)_\alpha P^\alpha
$$

$$
\begin{aligned}
\theta\omega\theta^{-1} &= \begin{pmatrix} \hat{\theta}^j_i & v_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega_j{}^k & \omega_{j0} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\theta}^T{}_k{}^l & -\hat{\theta}^T{}_k{}^m v_m \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \hat{\theta}^j_i & v_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega_j{}^k \hat{\theta}^l_k & -\omega_j{}^k \theta^m{}_k v_m + \omega_{j0} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \hat{\theta}^j_i \omega_j{}^k \hat{\theta}^l_k & -\hat{\theta}^j_i \omega_j{}^k \theta^m{}_k v_m + \hat{\theta}^j_i \omega_{j0} \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

Chapter II, Problem 4.

$$
\begin{aligned}
p^\mu p_\mu &= \bar{p}_\rho \Lambda^\rho{}_\mu \Lambda^\mu{}_\sigma \bar{p}^\sigma = \bar{p}_\rho \delta^\rho{}_\sigma \bar{p}^\sigma = \bar{p}_\rho \bar{p}^\rho \\
p^\mu p_\mu U &= \bar{p}_\rho \Lambda^\rho{}_\mu \Lambda^\mu{}_\sigma \bar{p}^\sigma U \\
&= (\bar{p}_\rho \Lambda^\rho{}_\mu)(\Lambda^\mu{}_\sigma \bar{p}^\sigma) U \\
&= (U \bar{p}_\mu U^{-1})(U \bar{p}_\mu U^{-1}) U \\
&= U \bar{p}_\mu \bar{p}^\mu \\
&= U p_\mu p^\mu
\end{aligned}
$$

The same is true for $w_\mu w^\mu$ provided w^μ transforms as a vector.

$$
\begin{aligned}
w_\mu &= \epsilon_{\mu\nu\rho\lambda} J^{\nu\rho} p^\lambda \\
&= \epsilon_{\mu\nu\rho\lambda} \bar{J}^{\alpha\beta} \Lambda_\alpha{}^\nu \Lambda_\beta{}^\rho (\bar{p}^\gamma \Lambda_\gamma{}^\lambda) \\
&= \epsilon_{\mu\nu\rho\lambda} (U \bar{J}^{\nu\rho} U^{-1})(U \bar{p}^\lambda U^{-1}) + \epsilon_{\mu\nu\rho\lambda} \Lambda_\alpha{}^\nu \Lambda_\beta{}^\rho (a^\alpha \bar{p}^\beta - a^\beta \bar{p}^\alpha) \Lambda_\gamma{}^\lambda \bar{p}^\gamma \\
&= U (U^{-1} \epsilon_{\mu\nu\rho\lambda} U) \bar{J}^{\nu\rho} \bar{p}^\lambda U^{-1} + \bar{\epsilon}_{\delta\pi\sigma\tau} \Lambda^\delta{}_\mu \Lambda^\pi{}_\nu \Lambda^\sigma{}_\rho \Lambda^\tau{}_\lambda \Lambda_\alpha{}^\nu \Lambda_\beta{}^\rho (a^\alpha \bar{p}^\beta - a^\beta \bar{p}^\alpha) \Lambda_\gamma{}^\lambda \bar{p}^\gamma \\
&= U \bar{\epsilon}_{\mu\nu\rho\lambda} \bar{J}^{\nu\rho} \bar{p}^\lambda U^{-1} + \bar{\epsilon}_{\delta\pi\sigma\tau} \Lambda^\delta{}_\mu \Lambda^\delta{}_\nu \delta^\pi{}_\alpha \delta^\sigma{}_\beta \delta_\gamma{}^\tau (a^\alpha \bar{p}^\beta - a^\beta \bar{p}^\alpha) \bar{p}^\gamma \\
&= U \bar{\epsilon}_{\mu\nu\rho\lambda} \bar{J}^{\nu\rho} \bar{p}^\lambda U^{-1} + \Lambda^\delta{}_\mu \bar{\epsilon}_{\delta\pi\sigma\tau} (a^\pi \bar{p}^\sigma - a^\sigma \bar{p}^\pi) \bar{p}^\tau \\
&= U \bar{w}_\mu U^{-1} + \Lambda^\delta{}_\mu \bar{\epsilon}_{\delta\pi\sigma\tau} a^\pi \bar{p}^\sigma \bar{p}^\tau - \Lambda^\delta{}_\mu \bar{\epsilon}_{\delta\pi\sigma\tau} a^\sigma \bar{p}^\pi \bar{p}^\tau \\
&= U \bar{w}_\mu U^{-1} + \Lambda^\delta{}_\mu \bar{\epsilon}_{\delta\pi\sigma\tau} a^\pi \bar{p}^\sigma \bar{p}^\tau - \Lambda^\delta{}_\mu \bar{\epsilon}_{\delta\sigma\pi\tau} a^\pi \bar{p}^\sigma \bar{p}^\tau \\
&= U \bar{w}_\mu U^{-1} + \Lambda^\delta{}_\mu \bar{\epsilon}_{\delta\pi\sigma\tau} a^\pi \bar{p}^\sigma \bar{p}^\tau + \Lambda^\delta{}_\mu \bar{\epsilon}_{\delta\pi\sigma\tau} a^\pi \bar{p}^\sigma \bar{p}^\tau \\
&= U \bar{w}_\mu U^{-1} + \Lambda^\delta{}_\mu \bar{\epsilon}_{\delta\pi\sigma\tau} a^\pi \bar{p}^\sigma \bar{p}^\tau + \Lambda^\delta{}_\mu \bar{\epsilon}_{\delta\pi\tau\sigma} a^\pi \bar{p}^\tau \bar{p}^\sigma \\
&= U \bar{w}_\mu U^{-1} + \Lambda^\delta{}_\mu \bar{\epsilon}_{\delta\pi\sigma\tau} a^\pi \bar{p}^\sigma \bar{p}^\tau + \Lambda^\delta{}_\mu \bar{\epsilon}_{\delta\pi\tau\sigma} a^\pi \bar{p}^\sigma \bar{p}^\tau \\
&= U \bar{w}_\mu U^{-1} + \Lambda^\delta{}_\mu \bar{\epsilon}_{\delta\pi\sigma\tau} a^\pi \bar{p}^\sigma \bar{p}^\tau - \Lambda^\delta{}_\mu \bar{\epsilon}_{\delta\pi\sigma\tau} a^\pi \bar{p}^\sigma \bar{p}^\tau \\
&= U \bar{w}_\mu U^{-1}
\end{aligned}
$$

Since w_μ transforms as a vector,

$$w_\mu w^\mu U = U w_\mu w^\mu.$$

Chapter II, Problem 5.

The "Lorentz Group" in question consists of the group of rotations in the 1,2 plane, plus the group of boosts parallel to the 1,2 plane, plus the group of translations in the 1,2 plane. An infinitesimal "Lorentz" transformation would be

$$\begin{aligned} U(1+\omega,\epsilon) &= 1 + i\omega_{12}J^{12} - i\epsilon_1 P^1 - i\epsilon_2 P^2 - i\epsilon_0 P^0 + i\omega_{10}J^{10} + i\omega_{20}J^{20} \\ &= 1 + i\omega_3 J^3 + i\omega_1 K^1 + i\omega_2 K^2 - i\epsilon_1 P^1 - i\epsilon_2 P^2 - i\epsilon_0 P^0 \end{aligned}$$

where

$$\begin{array}{ll} \omega_{12} = \omega_3 & J^{12} = J^3 \\ \omega_{10} = \omega_1 & J^{10} = K^1 \\ \omega_{20} = \omega_2 & J^{20} = K^2. \end{array}$$

the commutation relations are

$$\begin{array}{rclrcl} [K^i, K^i] &=& -i\epsilon^{ij3}J^3 & [J^3, K^i] &=& i\epsilon^{3ij}K^j \\ [K^i, P^j] &=& -iH\delta^{ij} & [J^3, H] &=& [P^i, H] = [H, H] = 0 \\ [K^i, H] &=& -iP^i & [J^3, P^i] &=& i\epsilon^{3ij}J^3 P^j. \end{array}$$

The pure translation group $T(1,a)$ forms a sub-group with the group multiplication rule

$$T(1+a)T(1+\bar{a}) = T(1, a+\bar{a})$$

so that finite transformations are represented by

$$U(1,a) = e^{-iP^\mu a_\mu}.$$

A rotation of θ in the 1,2 plane (R_θ) is similarly represented by

$$U(R_\theta, 0) = e^{iJ^3\theta}.$$

The one-particle states are defined as the states $\psi_{k,\sigma}$ which are simultaneous eigenvectors of $\mathbf{P}$, H and J^3 with respective eigenvalues 0, M and σ. σ can take any of the values 0, $\pm 1/2$, ± 1, $\pm 3/2$, etc. Since σ is a "Lorentz" invariant in $SO(2,1)$, define each different value of σ, positive or negative, as representing a different species of particle. A parity transformation in this space is equivalent to a rotation by $\theta = \pm\pi$, so the spin states are parity-invariant also:

$$P\psi_{k,\sigma} = \eta_\sigma \psi_{Pk,\sigma} = e^{\pm i\pi\sigma}\psi_{k,\sigma}$$

We must leave open the possibility that η_σ is different for each value of σ; i.e. each different species of particle.

To get to finite momentum states, apply $U(L(p))$ corresponding to the boost

$$
\begin{aligned}
L^\mu{}_\nu &= \begin{pmatrix} 1+(\gamma-1)\frac{p_1^2}{p_1^2+p_2^2} & (\gamma-1)\frac{p_1p_2}{p_1^2+p_2^2} & p_1\sqrt{\gamma^2-1} \\ (\gamma-1)\frac{p_1p_2}{p_1^2+p_2^2} & 1+(\gamma-1)\frac{p_2^2}{p_1^2+p_2^2} & p_2\sqrt{\gamma^2-1} \\ p_1\sqrt{\gamma^2-1} & p_2\sqrt{\gamma^2-1} & \gamma \end{pmatrix} \\
\gamma &= \frac{1}{M}\sqrt{p_1^2+p_2^2+m^2}
\end{aligned}
$$

$$
\begin{aligned}
P(L(p))P^{-1} &= R(\pi,0)L(p)R^{-1}(\pi,0) \\
&= L(R(\pi,0)p) = L(Pp) \qquad P\mathbf{p} = (-\mathbf{p}, \sqrt{p^2+m^2}) \\
P\psi_{p,\sigma} &= P\sqrt{\frac{M}{P_0}}U(L(p))\psi_{k,\sigma} \\
&= \sqrt{\frac{M}{P_0}}\{PU(L(p))P^{-1}\}P\psi_{k,\sigma} \\
&= \sqrt{\frac{M}{P_0}}U(P(L(p)P^{-1})e^{\pm i\pi\sigma}\psi_{k,\sigma} \\
&= \sqrt{\frac{M}{P_0}}U(L(Pp))e^{\pm i\pi\sigma}\psi_{k,\sigma} = e^{\pm i\pi\sigma}\psi_{Pp,\sigma}
\end{aligned}
$$

The effects of time-inversion on J^3, p^i and k^i are the same as in $SO(3,1)$:

$$TJ^3T^{-1} = -J^3 \quad Tp^iT^{-1} = -p^i \quad TK^iT^{-1} = K^i,$$

so that

$$Tp^i\psi_{k,\sigma} = (Tp^iT_{-1})T\psi_{k,\sigma} = -p^iT\psi_{k,\sigma} = 0,$$

remembering tht $\psi_{k,\sigma}$ is the standard zero-momentum one-particle state.

$$
\begin{aligned}
TH\psi_{k,\sigma} &= TM\psi_{k,\sigma} \\
(THT^{-1})T\psi_{k,\sigma} &= (TMT^{-1})T\psi_{k,\sigma} \\
H(T\psi_{k,\sigma} &= M(T\psi_{k,\sigma}) \\
T(J^3\psi_{k,\sigma}) &= (TJ^3T^{-1})(T\psi_{k,\sigma}) \\
(T\sigma\psi_{k,\sigma}) &= (T\sigma T^{-1})(T\psi_{k,\sigma}) = -J^3(T\psi_{k,\sigma}) \\
\sigma(T\psi_{k,\sigma}) &= -J^3(T\psi_{k,\sigma})
\end{aligned}
$$

so that

$$T\psi_{k,\sigma} = \zeta_\sigma\psi_{k,-\sigma}.$$

Parity reverses under time-inversion, so that a particle of spin $-\sigma$ is the time-reversed conjugate (anti-particle) of one of spin $+\sigma$. ζ_σ^* cannot be eliminated by a definition of particle states since ζ_σ is different for each species.

For finite momentum states,

$$
\begin{aligned}
TL(p)T^{-1} &= L(Pp) \\
T\psi_{p,\sigma} &= T\sqrt{\frac{M}{p_0}}U(L(p))\psi_{k,\sigma} \\
&= \left(T\sqrt{\frac{M}{p_0}}T^{-1}\right)\{TU(L(p))T^{-1}\}T\psi_{k,\sigma} \\
&= \sqrt{\frac{-M}{-p_0}}\{TU(L(p))T^{-1}\}T\psi_{k,\sigma} \\
&= \sqrt{\frac{M}{p_0}}U\{L(Pp)\}T\psi_{k,\sigma} \\
&= \sqrt{\frac{M}{p_0}}U\{L(Pp)\}\psi_{k,-\sigma}\zeta_\sigma = \zeta_\sigma\psi_{Pp,-\sigma}
\end{aligned}
$$

Chapter II, Problem 6.

Acting on a state $\psi_{k,\sigma}$ that is defined as an eigenvector of p^μ with an eigenvalue $k^\mu = (0, k, k)$ and an eigenvector of J^3 with an eigenvalue σ, the parity operator P yields a state with 4-momentum $(Pk)^\mu = (0, -k, -k)$ and $J^3 = \sigma$. Helicity (component of spin along the direction of motion) is undefined. As before, P is indistinguishable from a rotation in the 1,2 plane by $\theta = \pm\pi : P = R(\pm\pi, 0) = e^{i\pi J^3}$. Spin states are distinguished by integral or half-integral values of $\sigma : \sigma = 0, \pm 1/2, \pm 1$, etc. σ is invariant under all "Lorentz" transformations. States of different values of σ are defined to be different species of particle.

$$
\begin{aligned}
\psi_{p,\sigma} &= \sqrt{\frac{k}{p^0}}\, U\left(R(\mathbf{p})\right) B\left(\frac{|\mathbf{p}|}{k}\right) \psi_{k,\sigma} \\
P\psi_{p,\sigma} &= \sqrt{\frac{k}{p^0}} \left\{ PU\left(R(\mathbf{p}) B\left(\frac{|\mathbf{p}|}{k}\right)\right) P^{-1} \right\} P\psi_{k,\sigma} \\
&= \sqrt{\frac{k}{p^0}} \left\{ U\left(PR(\mathbf{p}) B\left(\frac{|\mathbf{p}|}{k}\right)\right) P^{-1} \right\} P\psi_{k,\sigma} \\
&= \sqrt{\frac{k}{p^0}}\, U\left\{PR(\mathbf{p})P^{-1}\right\} \left\{ PB\left(\frac{|\mathbf{p}|}{k}\right) P^{-1} \right\} \eta_\sigma \psi_{-k,\sigma} \\
PR(\mathbf{p})P^{-1} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -\cos\theta & \sin\theta & 0 \\ -\sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
PB\left(\frac{|\mathbf{p}|}{k}\right)P^{-1} &= \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \gamma & \sqrt{\gamma^2-1}\\ 0 & \sqrt{\gamma^2-1} & \gamma\end{pmatrix}\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\\
&= \begin{pmatrix}-1 & 0 & 0\\ 0 & -\gamma & -\sqrt{\gamma^2-1}\\ 0 & \sqrt{\gamma^2-1} & \gamma\end{pmatrix}\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\\
&= \begin{pmatrix}1 & 0 & 0\\ 0 & \gamma & \sqrt{\gamma^2-1}\\ 0 & \sqrt{\gamma^2-1} & \gamma\end{pmatrix}\\
P\psi_{p,\sigma} &= \sqrt{\frac{k}{p^0}}\,U\left(R(\mathbf{p})B\left(\frac{|\mathbf{p}|}{k}\right)\right)\eta_\sigma\psi_{-k,\sigma}\\
&= \eta_\sigma\psi_{-p,\sigma} = \eta_\sigma\psi_{Pp,\sigma}
\end{aligned}
$$

Spin is parity invariant. The phase η_σ must be regarded as different for each value of σ; i.e., different for each species of particle.

The time-reversal operator takes $k^\mu = (0,k,k)$ into $(Pk)^\mu = (0,-k,k)$ and σ into $-\sigma$.

$$
U(R(\pi,0))Tk^\mu = k^\mu.
$$

This anti-commutes with J^3, so

$$
\begin{aligned}
U(R(\pi))T\psi_{k,\sigma} &= +\zeta_\sigma\psi_{k,-\sigma}\\
U(R(\pi))T\psi_{p,\sigma} &= \sqrt{\frac{k}{p^0}}\,U\left(R(\pi)\right)TU\left(R(\mathbf{p})B\left(\frac{|\mathbf{p}|}{k}\right)\right)T^{-1}U^{-1}(R(\pi))U(R(\pi))U^{-1}T\psi_{k,\sigma}\\
&= \zeta_\sigma\sqrt{\frac{k}{p^0}}\,U\left(R(\pi)T(R(\mathbf{p}))B\left(\frac{|\mathbf{p}|}{k}\right)T^{-1}U^{-1}\left(R^{-1}(\pi)\right)\right)\psi_{k,-\sigma}\\
&= \zeta_\sigma\sqrt{\frac{k}{p^0}}\,U\left(R(\pi)R(\mathbf{p})T^{-1}TB\left(\frac{|\mathbf{p}|}{k}\right)T^{-1}R(-\pi)\right\}\psi_{k,-\sigma}\\
TR(\mathbf{p})T^{-1} &= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\\
&= \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
TB\left(\frac{|\mathbf{p}|}{k}\right)T^{-1} &= \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}1&0&0\\0&\gamma&\sqrt{\gamma^2-1}\\0&\sqrt{\gamma^2-1}&\gamma\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\\
&= \begin{pmatrix}1&0&0\\0&\gamma&\sqrt{\gamma^2-1}\\0&-\sqrt{\gamma^2-1}&-\gamma\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\\
&= \begin{pmatrix}1&0&0\\0&\gamma&\sqrt{\gamma^2-1}\\0&\sqrt{\gamma^2-1}&\gamma\end{pmatrix}\\
U(R(\pi))T\psi_{p,\sigma} &= \zeta_\sigma\sqrt{\frac{k}{p^0}}\,U\left(R(\pi)R(\mathbf{p})B\left(\frac{|\mathbf{p}|}{k}\right)R(-\pi)\right)\psi_{k,-\sigma}
\end{aligned}
$$

Since $B\left(\frac{|\mathbf{p}|}{k}\right)$ commutes with a rotation of $\pm\pi$,

$$
\begin{aligned}
U(R(\pi))T\psi_{p,\sigma} &= \zeta_\sigma\sqrt{\frac{k}{p^0}}\,U\left(R(\pi)R(\mathbf{p})R(-\pi)B\left(\frac{|\mathbf{p}|}{k}\right)\right)\psi_{k,-\sigma}\\
&= \zeta_\sigma\sqrt{\frac{k}{p^0}}\,U\left(R(\mathbf{p})B\left(\frac{|\mathbf{p}|}{k}\right)\right)\psi_{k,-\sigma}\\
T\psi_{p,\sigma} &= \zeta_\sigma\sqrt{\frac{k}{p^0}}\,U(R(-\pi))U\left(R(\mathbf{p})B\left(\frac{|\mathbf{p}|}{k}\right)\right)\psi_{k,-\sigma}\\
T\psi_{p,\sigma} &= \zeta_\sigma\sqrt{\frac{k}{p^0}}\,U\left(R(-\pi)\right)U\left(R(\mathbf{p})B\left(\frac{|\mathbf{p}|}{k}\right)\right)U(R(\pi))U(R(-\pi))\psi_{k,-\sigma}\\
&= \zeta_\sigma\sqrt{\frac{k}{p^0}}\,U\left(R(-\pi)R(\mathbf{p})B\left(\frac{|\mathbf{p}|}{k}\right)R(\pi)\right)\psi_{-k,-\sigma}\\
&= \zeta_\sigma\sqrt{\frac{k}{p^0}}\,U\left(R(\mathbf{p})B\left(\frac{|\mathbf{p}|}{k}\right)\right)\psi_{-k,-\sigma}\\
&= \sqrt{\frac{k}{p^0}}\,U\left(R(-\pi)R(\mathbf{p})B\left(\frac{|\mathbf{p}|}{k}\right)\right)\psi_{k,-\sigma}\\
&= \zeta_\sigma\sqrt{\frac{k}{p^0}}\,U\left(R(-\mathbf{p}))B\left(\frac{|\mathbf{p}|}{k}\right)\right)\psi_{k,-\sigma}\\
&= \zeta_\sigma\psi_{-\mathbf{p},-\sigma} = \zeta_\sigma\psi_{Pp,-\sigma}
\end{aligned}
$$

Chapter III, Problem 7

$$
\begin{aligned}
(\phi_{E'\mathbf{p}'j'l's'n'\sigma'}, \phi_{E\mathbf{p}jlsn\sigma}) &= \sum_{\sigma_1\sigma_2 n''} d^3\mathbf{p}_1\, d^3\mathbf{p}_2 (\phi_{E'\mathbf{p}'j'l's'n'\sigma'}, \phi_{E\mathbf{p}jlsn\sigma})(\phi_{E'\mathbf{p}'j'l's'n'\sigma'}, \phi_{\mathbf{p}_1\sigma_1\mathbf{p}_2\sigma_2 n''}) \times \\
&\qquad \left(\phi_{\mathbf{p}_1\sigma_1\mathbf{p}_2\sigma_2}, \phi_{E0jlsn\sigma}\right) \\
&= \sum_{\sigma_1\sigma_2 n''} \int d^3\mathbf{p}_1\, d^3\mathbf{p}_2 \left\{ \sqrt{\frac{E'}{|\mathbf{p}_1| E_1 E_2}}\, \delta^3(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2) \delta\left(E' - \sqrt{\mathbf{p}_1^2 + m_1^2}\right.\right. \\
&\quad \left.\times - \sqrt{\mathbf{p}_2^2 - m_2^2}\right) \delta_{n'n''} \\
&\quad \sum_{m',\mu'} C_{s_1 s_2}(s', \mu'; \sigma_1, \sigma_2) C_{l's'}(j', \sigma'; m', \mu') Y_{l'}^{*m'}(\mathbf{p}_1) \left(|\mathbf{p}_1| \frac{E_1 E_2}{E'}\right)^{-1/2} \delta^3(0 - \ \mathbf{p}_1 - \mathbf{p}_2)
\end{aligned}
$$