Recurrence Relations

Now we'll look at how we might solve recurrence relations.

1. A recurrence relation is like a differential equation. Its an expression describing a sequence of numbers where the value of some number in the sequence can be defined in terms of earlier numbers in the sequence.

2. We're already intimately familiar with the Fibonacci sequence, where the value of a number in the sequence is the sum of the previous two numbers. We'll write it as:

$$a_n = a_{n-1} + a_{n-2}$$

3. As a member of the research staff at the Center for Disease Control in Atlanta, you have been tasked with tracking the spread of the recently discovered Severe Acute Respiratory Syndrome (SARS). After talking to several health officials, you have decided to model SARS with the following assumptions:

   (a) A person with SARS is sick for four days.

   (b) A person spreads SARS to two other people for each day sick.

   (c) At day 0, there was only 1 person infected.

   In order to help analyze this, you built the following table:

	Number People Sick				total	
	1	2	3	4	number	
Day	day	days	days	days	sick	Comments
0	1	0	0	0	1	Poor Sucker
1	2	1	0	0	3	Spreads to Family
2	6	2	1	0	9	Uh Oh
3	18	6	2	1	27	Look out!
4	54	18	6	2	80	Original person dead!
5	160	54	18	6	238	Government begins cover-up
6	476	160	54	18	708	Back page of newspapers
7	1416	476	160	54	2106	Displaces war news

   By looking at this table, you discover the recurrence relations a_n describing the number of people sick on day n. You assumed that $\forall n, n < 0 \rightarrow a_n = 0$. That is $a_{-1} = a_{-2} = a_{-3} = \cdots = 0$. You also assumed $a_0 = 1$.

   What is the recurrence relation you discovered?

   $a_n = 3a_{n-1} - a_{n-4} + a_{n-5}$

4. Normally to find out the value of a_n for some particular n, we need to find out the value of all the numbers that came before. What a pain! We'll define "solving" a recurrence relation as coming up with a formula so that we can determine a_n without having to determine all the numbers that came before,

5. some recurrence relations are solvable using algebraic techniques, but they're often tricky, they require some math many of you don't know, and it still won't work for many relations.

6. Instead, we'll look at classes of recurrence relations that we know we can solve in a general way.

   (a) A recurrence relation is *homogeneous* if every term on the right-hand side contains a factor of the form a_k.

      i. $a_n = na_{n-1}a_{n-2} + 3a_0$ is homogeneous.

      ii. $a_n = na_{n-1}a_{n-2} + 3a_0 + 7$ is not.

   (b) A recurrence relation is *linear* if every term on the right-hand side contains no more than one factor of the form a_k.

      i. $a_n = na_{n-1}a_{n-2} + 3a_0$ is homogeneous, but not linear.

      ii. $a_n = na_{n-1} - a_{n-2} + 3a_0$ is both homogeneous and linear.

   (c) A recurrence relation has *constant coefficients* if n does not appear in any term, except in a subscript.

      i. $a_n = na_{n-1} - a_{n-2} + 3a_0$ is homogeneous and linear, but does not have constant coefficients.

      ii. $a_n = 4a_{n-1} - a_{n-2} + 3a_0$ is a linear homogeneous recurrence relation with constant coefficients.

7. Our first technique will only work on linear homogeneous recurrence relation with constant coefficients. It generates a solution of the form $a_n = r^n$ where r is a constant.

   (a) Let us consider an example we all know: calculating interest. If we keep our money (c dollars) in the bank at 11% annual interest, compounded annually, how much money do we have after n years? $a_n = 1.11a_{n-1}, a_0 = c$

   We know a formula for this, don't we? $a_n = 50(1.11)^n$.

   (b) The nice thing is that LHRRw/CC will always be of that form, so:

   (c) Suppose we have a LHRRw/CC: $a_n = c_1a_{n-1} + c_2a_{n-2} + ... + c_ka_{n-k}$. If $a_n = r^n$, then we could substitute:

$$r^n = c_1r^{n-1} + c_2r^{n-2} + ... + c_kr^{n-k}$$

$$r^k = c_1r^{k-1} + c_2r^{k-2} + ... + c_kr^{k-k} \text{ divide by} r^{n-k}$$

$$r^k - c_1r^{k-1} - c_2r^{k-2} - ... - c_kr^{k-k} = 0$$

(d) If we can find a solution for this equation (called the characteristic equation) in terms of r, then we know the value of r in the original equation $a_n = r^n$

(e) But this isn't the whole story. Take the case of calculating interest. $a_n = 1.11a_{n-1}$. The characteristic equation is: $r - 1.11 = 0$. The solution to that is obviously $r = 1.11$, implying that $a_n = 1.11^n$. Which is true, this is a solution (sort of) to the recurrence relation $a_n = 1.11a_{n-1}$. What's it missing? a_0! We need to account for that.

(f) We account for a_0 by including a constant α in the solution: $a_n = \alpha r^n$, and then solving for α based on the value of a_0. For example, $a_n = \alpha 1.11^n$, plus $a_0 = 50$, gives us $\alpha = 50$, and the final solution $a_n = 1.11^n 50$, which is the interest equation you know and love.

(g) How do we know that this technique works? We prove this theorem from the text: "Let C be a real number. Suppose that r-C=0 is the characteristic equation with one real root r. Then, the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = ca_{n-1}$ if and only if $a_n = \alpha r^n$ where α is a constant."

(h) The proof in the text is hard to follow, so it is worth going over it in detail to understand how the proof stuff you learned before actually applies to a real proof.

(i) Break the theorem into propositions as follows:

"A:(Let C be a real number. Suppose that r-C=0 is the characteristic equation with one real root r.) Then, B:(the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = ca_{n-1}$) if and only if C:($a_n = \alpha r^n$ where α is a constant.)"

(j) This means the theorem says "if A, then (B if and only if C)" or $A \to (B \leftrightarrow C)$.

$A \to ((B \to C) \wedge (C \to B))$ Since $(X \leftrightarrow Y)$ is $(X \to Y) \wedge (Y \to X)$
$(A \to (B \to C)) \wedge (A \to (C \to B))$ Since $X \to (Y \wedge Z)$ is $(X \to Y) \wedge (X \to Z)$
$((A \wedge B) \to C) \wedge ((A \wedge C) \to B))$ Since $X \to (Y \to Z)$ is $(X \wedge Y) \to Z$

(k) So this tells us we need to prove 2 things:

"IF C is a real number, r-C=0 is the characteristic equation with one real root r, and the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = ca_{n-1}$, THEN $a_n = \alpha r^n$ where α is a constant."

and

"IF c is a real number, r-c=0 is the characteristic equation with one real root r, and $a_n = \alpha r^n$ where α is a constant, THEN the sequence $\{a_n\}$ is a solution of the recurrence relation."

Lets do the second one first: prove the sequence $\{a_n\}$ is a solution of the recurrence relation. Assumptions:

$$r - c = 0$$
$$r = c \tag{1}$$
$$a_n = \alpha r^n \tag{2}$$

One way to show that $\{a_n\}$ is a solution of is to start with $a_n = ca_{n-1}$, and come up with a_n:

$$\begin{aligned} ca_{n-1} &= c\alpha r^{n-1} \text{ by (2)} \\ &= r\alpha r^{n-1} \text{ by (1)} \\ &= \alpha r^n \\ &= a_n \text{ by (2)}\square \end{aligned}$$

OK, time to do the second part: prove $a_n = \alpha r^n$ where α and a_0 are constants. A reminder: a RR w/o starting points has multiple solutions.

First we find an α that is consistent with the starting point. We do this by looking at: $a_0 = \alpha r^0 = \alpha$. Thus we know that $\alpha = a_0$. This next part is tricky:

i. We know from the first part that *if* $a_n = \alpha r^n$, *then* the sequence generated by that equation, $\{a_n\}$, is *a* solution.

ii. Now, assume some sequence of numbers is a solution to the RR. Assume we take some equation of the form $a_n = \alpha r^n$ and match it to any part of the particular sequence (that is, that equation generates that portion of the sequence).

iii. from 7(k)i, we know that $a_n = \alpha r^n$ must be *a* solution, and since $a_n = \alpha r^n$, with $\alpha = a_0$ is a *particular* solution and all sequences are unique, it must be the *only* solution.

(l) What does this say, anyway? Well, tells us 3 things: we can solve $a_n = ca_{n-1}$ with $a_n = \alpha r^n$, solve for α using the a_0, and that using this technique gives us ALL solutions. These are 3 very important things.

(m) We're already seen this theorem in action, with our compound interest case.

8. But what about more complicated recurrence relations, such as those of degree 2?

(a) We can apply the same proof above to those cases:

"IF c_1 and c_2 are real numbers, $r^2 - c_1 r - c_2 = 0$ is the characteristic equation with two real roots r_1 and r_2 of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, and the sequence $\{a_n\}$ is a solution of the recurrence relation, THEN $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ where α_1 and α_2 are constants."

and

"IF c_1 and c_2 are real numbers, $r^2 - c_1 r - c_2 = 0$ is the characteristic equation with two real roots r_1 and r_2 of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, and $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ where α_1 and α_2 are constants, THEN the sequence $\{a_n\}$ is a solution of the recurrence relation."

(b) We prove these in the same ways, starting with the second:

$$
\begin{aligned}
a_n &= \alpha_1 r_1^n + \alpha_2 r_2^n && (3)\\
r^2 - c_1 r - c_2 &= 0 && (4)\\
r_1^2 &= c_1 r_1 + c_2 \text{ by } (4) && (5)\\
r_2^2 &= c_1 r_2 + c_2 \text{ by } (4) && (6)\\
c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \text{ by } (3)\\
&= c_1\alpha_1 r_1^{n-1} + c_1\alpha_2 r_2^{n-1} + c_2\alpha_1 r_1^{n-2} + c_2\alpha_2 r_2^{n-2}\\
&= c_1\alpha_1 r_1^{n-1} + c_2\alpha_1 r_1^{n-2} + c_1\alpha_2 r_2^{n-1} + c_2\alpha_2 r_2^{n-2}\\
&= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2)\\
&= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \text{ by (5)and (6)}\\
&= \alpha_1 r_1^n + \alpha_2 r_2^n\\
&= a_n \text{ by (3).}
\end{aligned}
$$

(c) Then we need to find the αs.

$$
\begin{aligned}
a_0 &= C_0 = \alpha_1 + \alpha_2\\
a_1 &= C_1 = \alpha_1 r_1 + \alpha_2 r_2\\
\alpha_2 &= C_0 - \alpha_1\\
C_1 &= \alpha_1 r_1 + (C_0 - \alpha_1) r_2\\
\alpha_1 &= \frac{C_1 - C_0 r_2}{r_1 - r_2}\\
\alpha_2 &= \frac{C_0 r_1 - C_1}{r_1 - r_2}
\end{aligned}
$$

(d) Note that this proof doesn't work if $r_1 = r_2$, both roots are the same, since we would be dividing by zero.

(e) How does this work? Lets try an example: $a_0 = -2$, $a_1 = 3$ and $a_n = a_{n-1} + 6a_{n-2}$. $k = 2$ so the characteristic equation is $r^2 - r - 6$. The roots of this are -2 and 3,

thus the solution is of the form $a_n = \alpha_1(-2)^n + \alpha_2 3^n$. We know:

$$\begin{aligned}
-2 &= \alpha_1(-2)^0 + \alpha_2 3^0 \text{ and,} \\
3 &= \alpha_2(-2)^1 + \alpha_2 3^1 \text{ so,} \\
-2 &= \alpha_1 + \alpha_2 \\
3 &= -2\alpha_1 + 3\alpha 2 \\
\alpha_1 &= -2 - \alpha_2 \\
3 &= -2(-2 - \alpha_2) + 3\alpha 2 \\
3 &= 4 + 2\alpha_2 + 3\alpha 2 \\
\alpha_2 &= -\frac{1}{5} \\
-2 &= \alpha_1 - \frac{1}{5} \\
\alpha_1 &= \frac{1}{5} - 2 \\
\alpha_1 &= -\frac{9}{5} \\
a_n &= -\frac{9}{5}(-2)^n - \frac{1}{5}3^n
\end{aligned}$$

(f) Cute, huh? What about Mr. Fibonacci? $a_n = a_{n-1} + a_{n-2}$, $a_0 = 0$ and $a_1 = 1$. Again $k = 2$ and the characteristic equation is $r^2 - r - 1 = 0$. This is not factorable, but we can find the roots using the quadratic formula: $r_1 = \frac{1+\sqrt{5}}{2}$ $r_2 = \frac{1-\sqrt{5}}{2}$ so $a_n = \alpha_1(\frac{1+\sqrt{5}}{2})^n + \alpha_2(\frac{1-\sqrt{5}}{2})^n$. $\alpha_1 = 1/\sqrt{5}$, $\alpha_2 = -1/\sqrt{5}$.

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

9. As I said, this doesn't work if the two roots are the same- we end up dividing by zero. We need a different solution if we have that case. It turns out that, if there is one distinct root of the characteristic equation $r^2 - c_1 r - c_2 = 0$, then $a_n = \alpha_1 r^n + \alpha_2 n r^n$

   Try: $a_n = 2a_{n-1} - a_{n-2}$, $a_0 = 2, a_1 = 1$.

   $r^2 - 2r + 1 = 0$, so the unique root is $r = 1$. Thus, $\alpha_1 = 2, \alpha_2 = -1$ and $a_n = 2 - n$.

10. That about wraps up recurrence relations of degree 2, but what about higher degrees?

    (a) If a characteristic equation $r^k - c_1 r^{k-1} - \cdots - c_k = 0$ has k roots, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$.

    (b) For example, given the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2} - 12a_{n-3}$ and $a_0 = 6, a_1 = 1, a_2 = 29$ what is the solution?

    (c) The characteristic equation is: $r^3 - 3r^2 - 4r + 12 = 0$ which has roots:$(r_1, r_2, r_3) = (3, 2, -1)$. Thus $a_n = \alpha_1 3^n + \alpha_2 2^n + \alpha_3(-2)^n$. And we know $\alpha_1 + \alpha_2 + \alpha_3 = 6$ and $3\alpha_1 + 2\alpha_2 - 2\alpha_3 = 1$ and $\alpha_1 3^2 + \alpha_2 2^2 + \alpha_3(-2)^2 = 29$ which gives us:

$\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3$, for a final solution of:

$$a_n = 3^n + 2 \cdot 2^n + 3(-2)^n$$

11. OK, now what about the case of a LHRRw/CC of arbitrary degree where the number of roots of the characteristic equation is less than the degree?

    (a) suppose a characteristic equation $r^k - c_1 r^{k-1} - \cdots - c_k = 0$ has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$ (this means that root r_p occurs m_p times). The solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ is:

$$\begin{aligned} a_n =& (\alpha_0 + \alpha_1 n^1 + \alpha_2 n^2 + \cdots + \alpha_{m_1-1} n^{m_1-1}) r_1^n + \\ & (\beta_0 + \beta_1 n^1 + \beta_2 n^2 + \cdots + \beta_{m_2-1} n^{m_2-1}) r_2^n + \\ & (\delta_0 + \delta_1 n^1 + \delta_2 n^2 + \cdots + \delta_{m_3-1} n^{m_3-1}) r_3^n + \\ & \cdots \\ & (\omega_0 + \omega_1 n^1 + \omega_2 n^2 + \cdots + \omega_{m_t-1} n^{m_t-1}) r_t^n \end{aligned} \tag{7}$$

    (b) wow.

    (c) For example: if $a_n = 10a_{n-1} - 40a_{n-2} + 82a_{n-3} - 91a_{n-4} + 52a_{n-5} - 12a_{n-6}$ then the characteristic equation is: $r^6 - 10r^5 + 40r^4 - 82r^3 + 91r^2 - 52r + 12 = 0$. This factors to: $(r-2)(r-2)(r-1)(r-1)(r-1)(r-3) = 0$ for roots of $(r_1, r_2, r_3) = (2, 1, 3)$ and multiplicities of $(m_1, m_2, m_3) = (2, 3, 1)$. Then, by (7):

$$\begin{aligned} a_n =& (\alpha_0 + \alpha_1 n^1) r_1^n + \\ & (\beta_0 + \beta_1 n^1 + \beta_2 n^2) r_2^n + \\ & (\delta_0) r_3^n \end{aligned}$$

    Then all we would have to do is solve for the various constants using the initial values $a_0, a_1, a_2, a_3, a_4, a_5$.

12. Linear **NON-HOMOGENEOUS** Recurrence Relations with Constant Coefficients.