Inverse Kinematics

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Once we have a mathematical model of where the robot's hand is given the position of the motors (via the angles of the joints) we can begin to ask the real question of what are the joint angles, and thus the motor positons.

- We can think of the problem as, given a position, how do we get here, if we're constrained to getting there by following along the links of the robot.
- We can relax this question, and come up with the easier, "how could we get there if we could move in any direction we want, and then orient ourselves properly.
- Yes, that's not the same thing, but its related, it shows us a good technique, and it has the side effect of giving us useful information.
  - given a T:

$$\begin{bmatrix} x_x & y_x & z_x & p_x \\ x_y & y_y & z_y & p_y \\ x_z & y_z & z_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

    calculated by the product of A's,
  - recall that T is also the product of translation and rotation from the reference frame:
    $T = Trans(x, y, z)Rot(z, \phi)Rot(y, \beta)Rot(x, \psi)$.
    which is:

$$\begin{bmatrix} \cos(\phi)\cos(\beta) & \cos(\phi)\sin(\beta)\sin(\psi) - \sin(\phi)\cos(\psi) & \cos(\phi)\sin(\beta)\cos(\psi) + \sin(\phi)\sin(\psi) & p_x \\ \sin(\phi)\cos(\beta) & \sin(\phi)\sin(\beta)\sin(\psi) + \cos(\phi)\cos(\psi) & \sin(\phi)\sin(\beta)\cos(\psi) - \cos(\phi)\sin(\psi) & p_y \\ -\sin(\beta) & \cos(\beta)\sin(\psi) & \cos(\beta)\cos(\psi) & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - This matrix describes the process of moving directly to a location in space, and then applying yaw, pitch, and roll, as if we had a magic robot.
  - Once we have this, we set our 2 versions of T equal to each other and solve for the position p as well as ϕ, β, and ψ. Why can we do this? Becaouse both matrices

describe the same location!

$$\beta = \arcsin(-x_z)$$
$$\psi = \arcsin(\frac{y_z}{\cos(\beta)})$$
$$\phi = \arcsin(\frac{x_y}{\cos(\beta)})$$

- And p is just the rightmost column.

- This has the nice property of telling us what the yaw pitch and roll are. That's nice because it turns the ugly upper lefthand 3×3 matrix into something we can understand.
- It has the unfortunate property of potential innaccuracies:
  - arcsin is innaccurate when β is $\pi/2, \ldots$ etc. Small variations in the input result is large changes in the angle.
  - Meanwhile, when β is near $\pi/2, 3\pi/2, \ldots$ etc, the cosine gets close to 0, making those fractions problematic.
- In general, we like to use arctangent when we can calculate both the sine and cosine parts:
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  - We calculate arctan with 2 arguments, the sin part and the cos part. These correspond to y and x on the unit circle, right?
  - We assume the range for arctan is $-\pi$ to π
  - We use the following cases:

  1. $x = 0$, $y > 0$. Return $\pi/2$
  2. $x = 0$, $y < 0$. Return $-\pi/2$
  3. $y = 0$, $x > 0$. Return 0
  4. $y = 0$, $x < 0$. Return $-\pi$
  5. $\frac{xy}{|xy|} \arctan \left|\frac{y}{x}\right|$

- OK, so how do we find the θs? The same technique as above, but we start with one matrix that is a desired position, and another that is the result of multiplying all the A matrices together. That second matrix still has the θs as unknowns. Thus it is really ugly.
  - Let's pretend we have a robot with just the first 2 links of your arms. In that case, we have:

$$\begin{bmatrix} \cos(\Theta_1) & -\sin(\Theta_1) & 0 & l_1\cos(\Theta_1) \\ \sin(\Theta_1) & \cos(\Theta_1) & 0 & l_1\sin(\Theta_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\Theta_2) & 0 & \sin(\Theta_2) & l_2\cos(\Theta_2) \\ \sin(\Theta_2) & 0 & -\cos(\Theta_2) & l_2\sin(\Theta_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

(the 2nd matrix looks odd because it is the application of both the rotation around z and the rotation around x present in the second link)

$$\begin{bmatrix} C_1C_2 - S_1S_2 & 0 & S_2C_1 + C_2S_1 & l_1C_1 + l_2C_1C_2 - l_2S_1S_2 \\ S_1C_2 + C_1S_2 & 0 & S_1S_2 - C_1C_2 & l_1S_1 + l_2S_1C_2 + l_2C_1S_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_x & y_x & z_x & p_x \\ x_y & y_y & z_y & p_y \\ x_z & y_z & z_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Maybe not immediately illucidating, but maybe some trig identities will help us:

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha+\beta)$$
$$\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta)$$

We'll represent $\cos(\Theta_1 + \Theta_2)$ as C_{12}, yielding:

$$\begin{bmatrix} C_{12} & 0 & S_{12} & l_1C_1 + l_2C_{12} \\ S_{12} & 0 & -C_{12} & l_1S_1 + l_2S_{12} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_x & y_x & z_x & p_x \\ x_y & y_y & z_y & p_y \\ x_z & y_z & z_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- How does this help? Well, lets start picking out individual equations

$$\begin{aligned} C_{12} &= x_x \\ l_1C_1 + l_2C_{12} &= p_x \\ l_1C_1 + l_2x_x &= p_x \\ C_1 &= \frac{p_x - l_2x_x}{l_1} \\ \Theta_1 &= \arccos\frac{p_x - l_2x_x}{l_1} \end{aligned}$$

- Recall that we want arctan as a function of both cosine and sine...

$$\begin{aligned} S_{12} &= x_y \\ l_1S_1 + l_2S_{12} &= p_y \\ l_1S_1 + l_2x_y &= p_y \\ S_1 &= \frac{p_y - l_2x_y}{l_1} \\ \Theta_1 &= \arctan\left(\frac{p_y - l_2x_y}{l_1}, \frac{p_x - l_2x_x}{l_1}\right) \\ \Theta_1 + \Theta_2 &= \arctan\left(S_{12}, C_{12}\right) \\ &= \arctan\left(x_y, x_x\right) \\ \Theta_2 &= \arctan\left(x_y, x_x\right) - \Theta_1 \end{aligned}$$

- Is that all there is? Yes, and no. That is the basic technique, but there is one more little trick you'll need to know.