Chris Brown, January 2008

Proofs of bounds on the "hop" algorithm for searching worms.

Note: Two pages prove # hops for a worm of lenght l is $\Omega(\sqrt{(l)})$ and $O(l^{2/3})$.

1. Consider a worm of length l.

2. Note than in an infinite grid, the number of squares at distance $d > 0$ from a given square is $4d$.

3. Consider a traversal of a worm made by the "hop" algorithm in searching for a square not actully in the worm. If n is the longest hop in a traversal and H the number of hops, we have that

$$H \leq \sum_{d=1}^{n} 4d = 4\left(\frac{n(n+1)}{2}\right) = 2n^2 + 2n,$$

because you can't have more hops of length d than there are squares at distance d from the target (x, y), and there are exactly $4d$ squares at distance d from (x, y). Similarly, the worm body length satisfies

$$l \geq \sum_{d=1}^{n} 4d \cdot d = 4\sum_{d=1}^{n} d^2 = \Theta(n^3),^{1} \text{ i.e. } l = \Omega(n^3).$$

4. **Theorem:** In an H-hop traversal, the longest hop, n, satisfies

$$n \geq \left\lceil \frac{-1+\sqrt{1+2H}}{2} \right\rceil.$$

Proof: From the above, we know that $H \leq 2n^2 + 2$, which we can rewrite as $n^2 + n - H/2 \geq 0$. Of course we're only interested in positive values of n, and by the quadratic formula the only positive root of $n^2 + n - H/2$ is $\frac{-1+\sqrt{1+2H}}{2}$ and therefore $n \geq \frac{-1+\sqrt{1+2H}}{2}$. Finaly, since n is an integer, we get $n \geq \left\lceil \frac{-1+\sqrt{1+2H}}{2} \right\rceil$.

5. Combining the two previous results, $l = \Omega(n^3)$ and $\lceil(-1+\sqrt{1+2H})/2\rceil$ we get

$$l = \Omega\left(\left(\frac{-1+\sqrt{1+2H}}{2}\right)^3\right) = \Omega\left(H^{3/2}\right).$$

6. Thus, by the def. of Ω, for some constant $a > 0$, when H is large we have

$$l \geq aH^{3/2} \text{ which means } (1/a)^{2/3}\, l^{2/3} \geq H \text{ which means } H = O\left(l^{2/3}\right).$$

[1] See page 1060 of the textbook.

Consider a worm that is wrapped around cell (x, y), so that it spirals away. We should describe this a bit precisely. Let a **d-square** be the square of cells (i, j) formed by the rows $j = x + d$ and $j = x - d$, and the columns $i = x + d$ and $i = x - d$. The worm starts at cell $(x, y - 1)$ and moves counter-clockwise around the 1-square until it uses up all the cells in the 1-square. Then it crosses into the 2-square and moves counter-clockwise around the 2-square until it uses up all those squares. We'll call this a "spiral worm".

Theorem: The "hop" algorithm makes $\Theta(\sqrt{n})$ hops in traversing the spiral worm around the point (x, y) it is searching for.

Proof: First note that a d-square consists of $8d$ cells. This may take a bit of thinking to convince yourself of. Next note the distance of any cell in a d-square from (x, y) is between d and $2d$.

Consider your last hop from a cell in a d square. The hop length is at most $2d$. Since that is less than the $8d$ cells that make up the $(d + 1)$-square you move into, you end your hop in the $(d + 1)$-square (as opposed to moving all the way around and out of it during that singel hop). So, if you land in a d-square, you will eventually land in a $(d + 1)$-square. Since our first hop lands in a 1-square, induction tells us that we land in every d-square the worm fills up.

So we have at least one landing in every d-square filled up by the worm, and clearly no more than 8, since each jump has length at least d and there are only $8d$ cells in the d-square. If the worm fills up the d-squares fromm $d = 1..k$. Then the number of hops, H, satisfies

$$k \leq H \leq 8k + 7, \quad ^{2}$$

i.e. $H = \Theta(k)$.. Meanwhile, the worm body has length l, where

$$l = \sum_{d=1}^{k} 8d + \text{spillover} = 4k(k + 1) + \text{spillover}$$

which means $k = \Theta(\sqrt{l})$. We have $H = \Theta(k)$ and $k = \Theta(\sqrt{l})$, so $H = \Theta(\sqrt{l})$.

Since we proved the number of hops is $\Theta(\sqrt{l})$ for a particular worm configuration, i.e. the spiral, the worst case can't be better and we get that the "hop" algorithm is $\Omega(\sqrt{(}n)$.

[2] The +7 is because the worm might fill some, though not all, of the $(k + 1)$-square.