CADs More Formally

- There are three basic operations for CAD: *Projection*, *Stack Construction*, and *Solution Formula Construction*.

- There are other operations as well, like the *truth propagation* we used for quantifier elimination, CAD *simplification*, and *adjacency computations*.

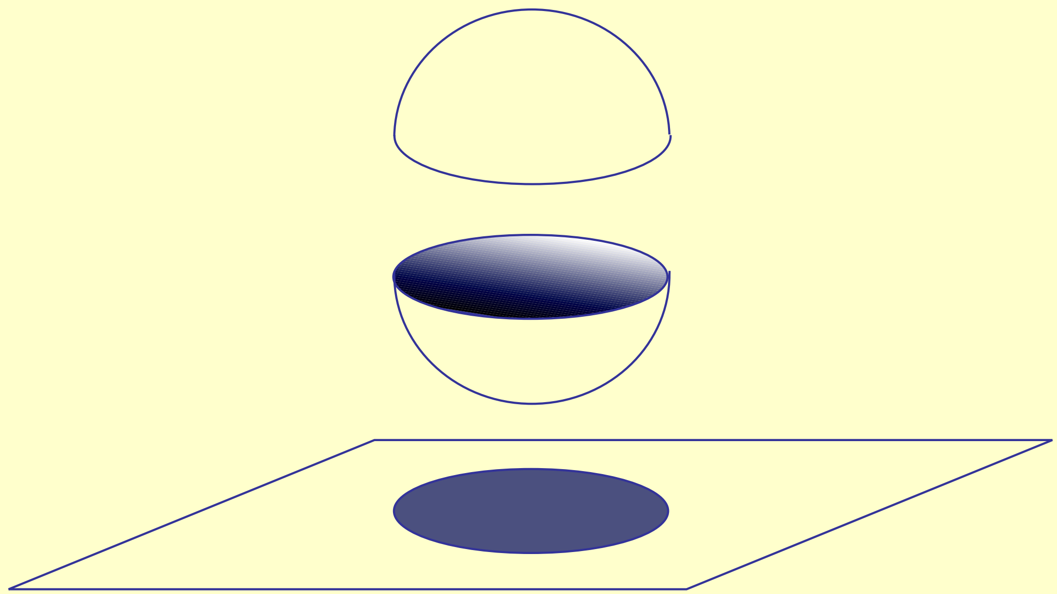
- This section focuses on the three basic operations.

Definition of *cylindrical*

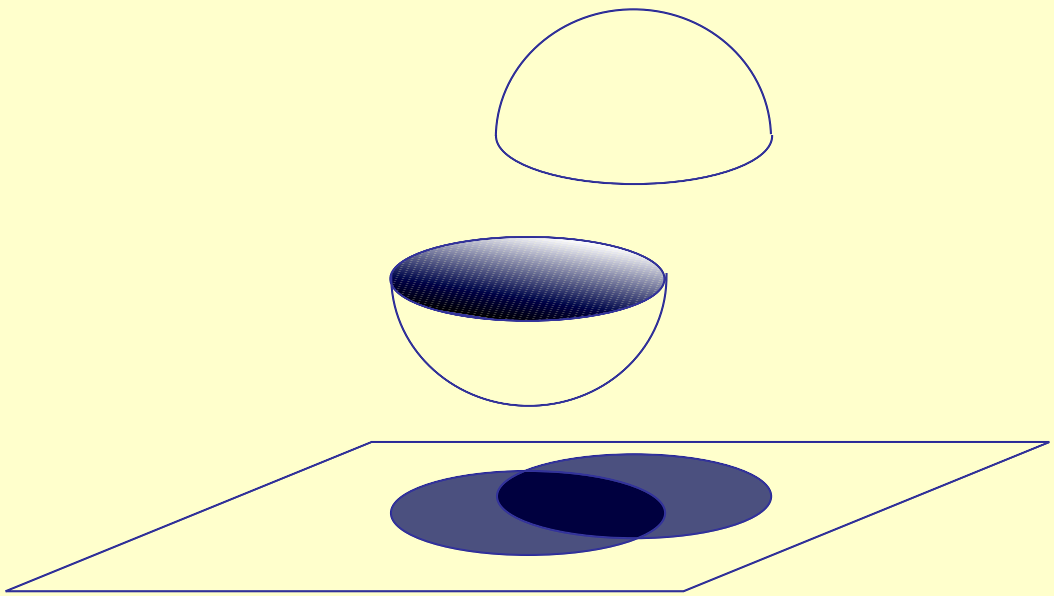
- **Definition:** A decomposition of $\mathbb{R}^n$ into finitely many connected regions is *cylindrical* if for any two partition regions a and b and for any k, where $0 < k < n$, the projections onto $\mathbb{R}^k$ of a and b are either identical or disjoint.

Cylindrically arranged sets

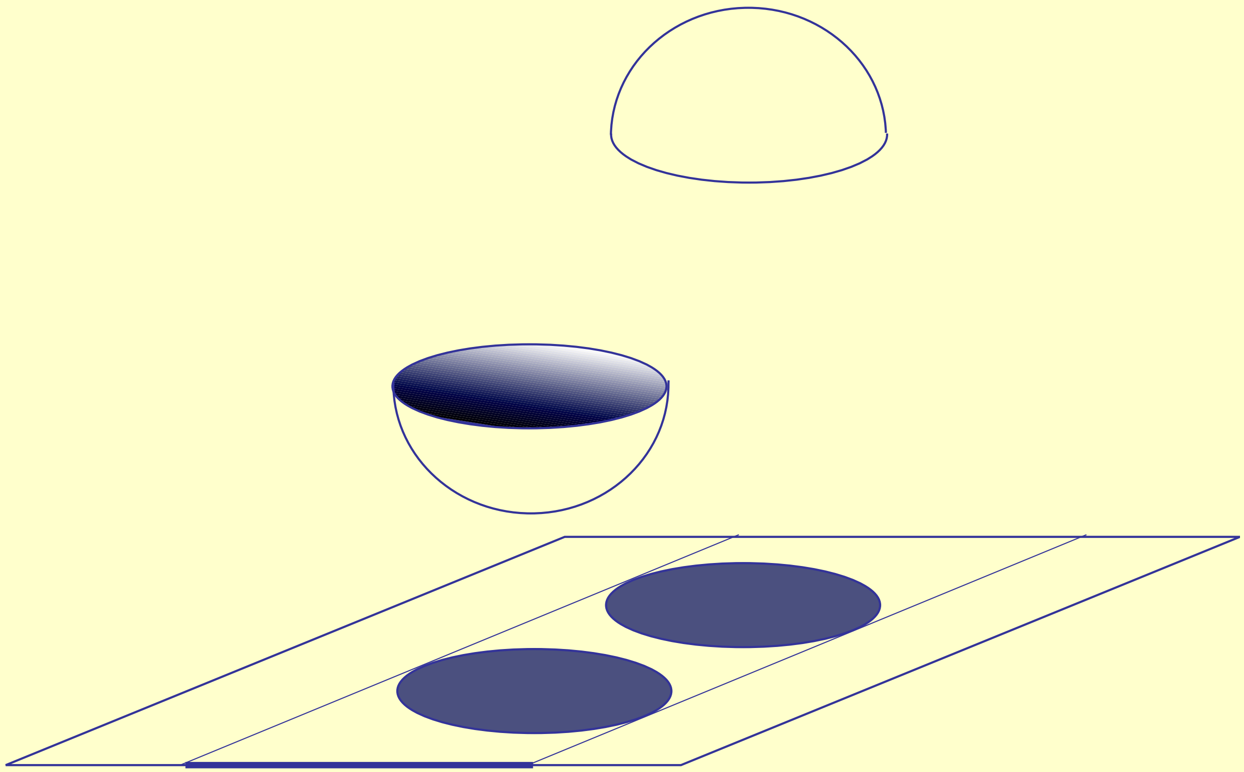
Cylindrically arranged sets



Cylindrically arranged sets



Cylindrically arranged sets



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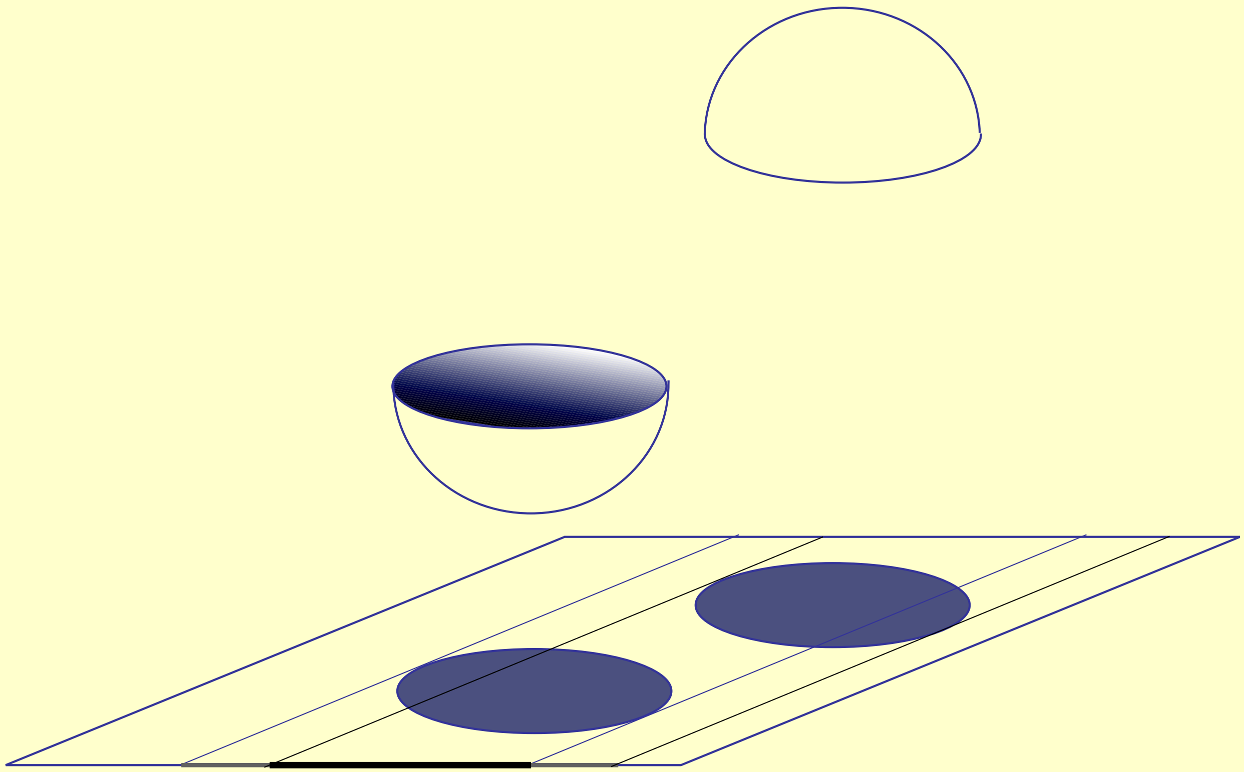


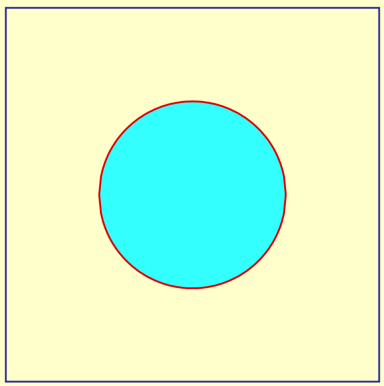
Definition of CAD

- **Definition:** A decomposition of $\mathbb{R}^n$ into finitely many connected regions is *cylindrical* if for any two partition regions a and b and for any k, where $0 < k < n$, the projections onto $\mathbb{R}^k$ of a and b are either identical or disjoint.

- **Definition:** A *Cylindrical Algebraic Decomposition* is cylindrical decomposition of $\mathbb{R}^n$ into semi-algebraic sets.

- **Definition:** A set of irreducible polynomials is a *projection factor set* if the natural algebraic decomposition it defines is a CAD.

How to make a decomposition cylindrical



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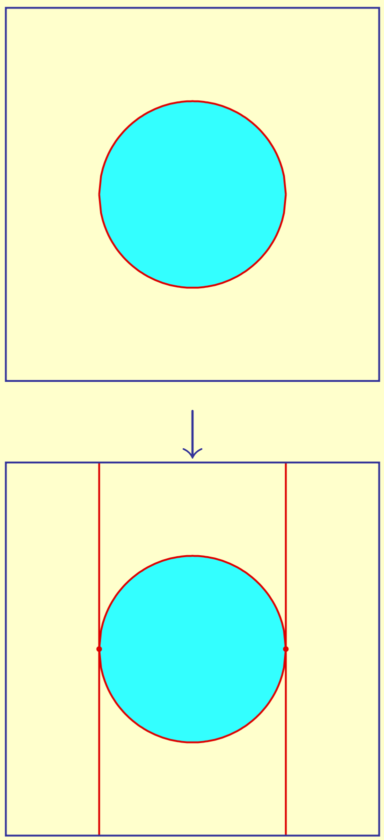
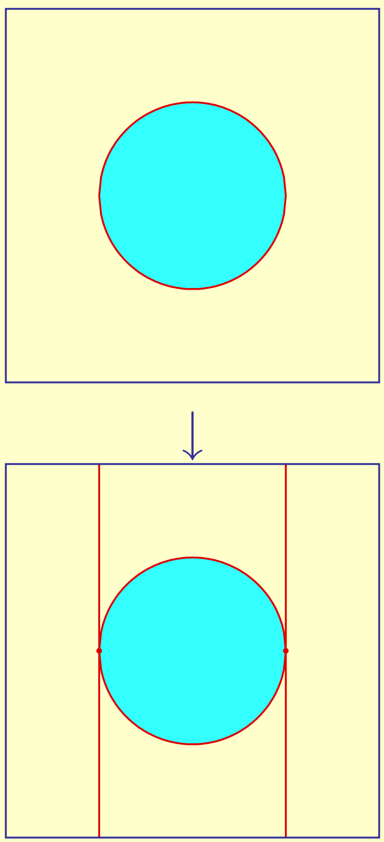
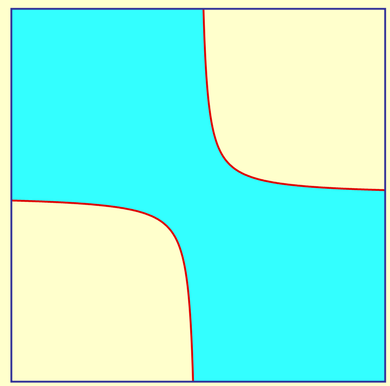


How to make a decomposition cylindrical





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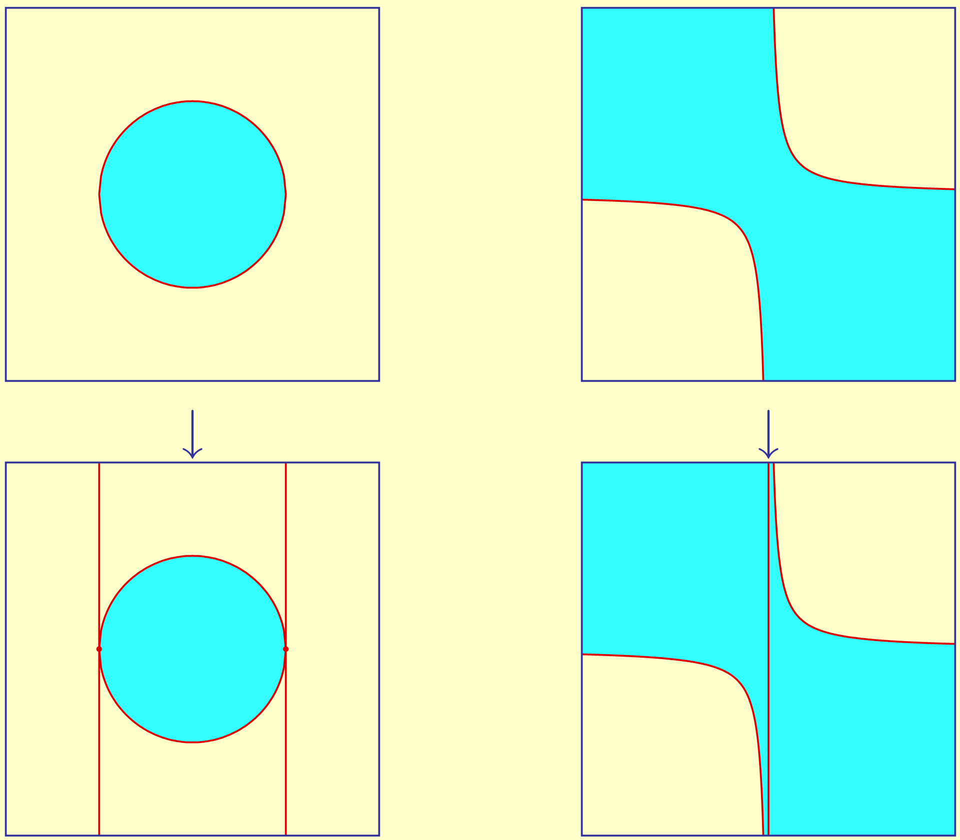

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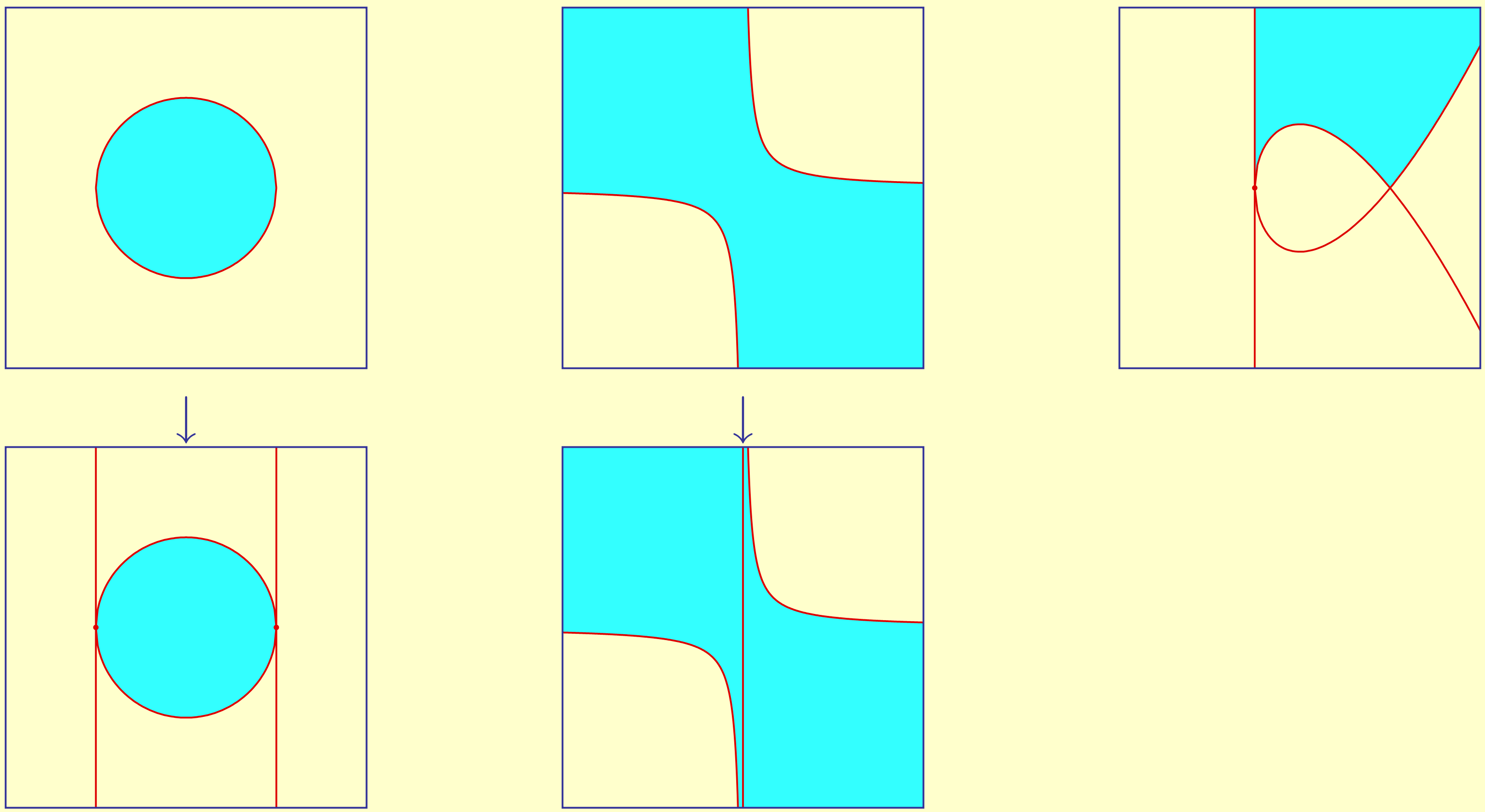

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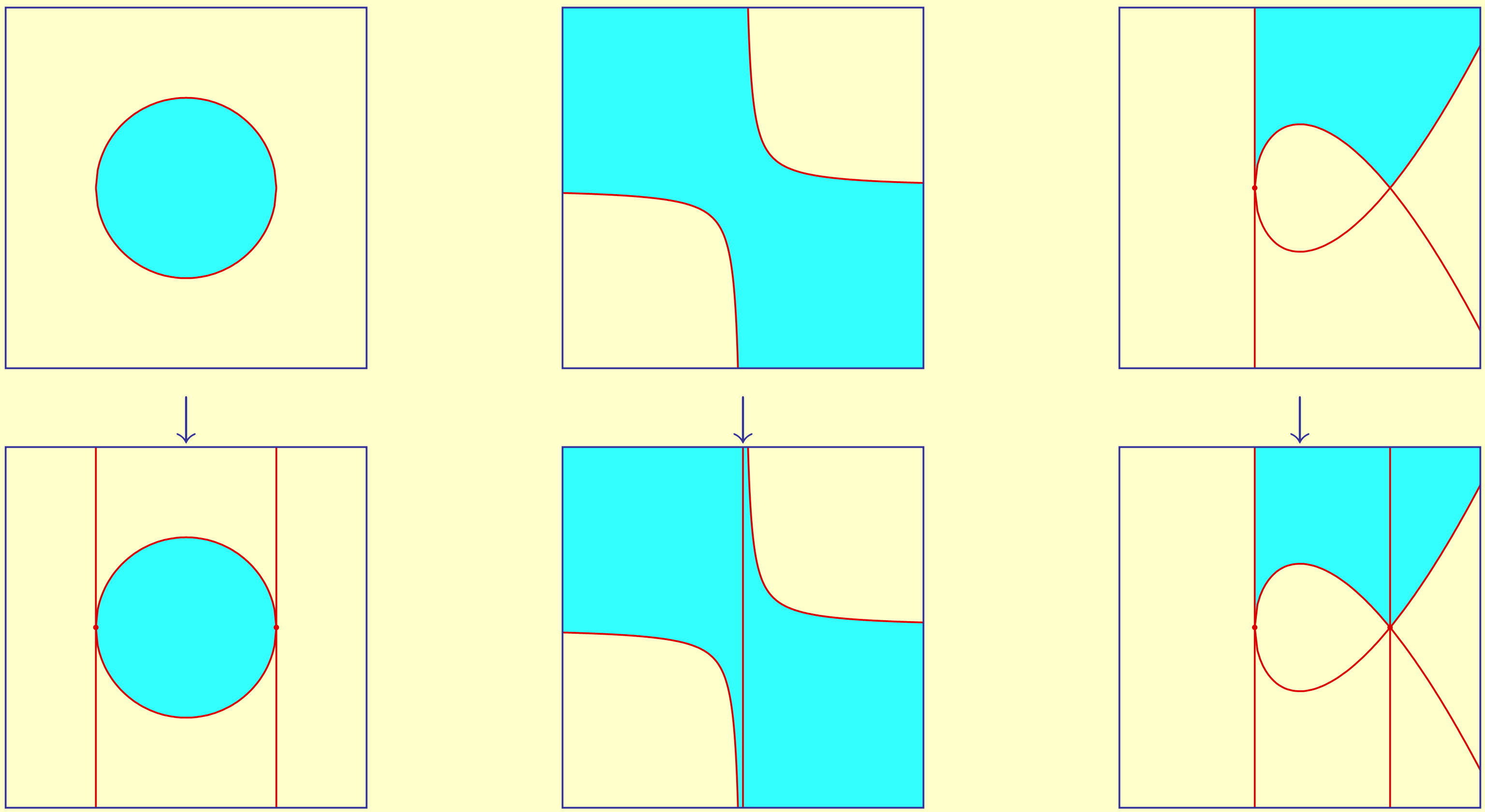
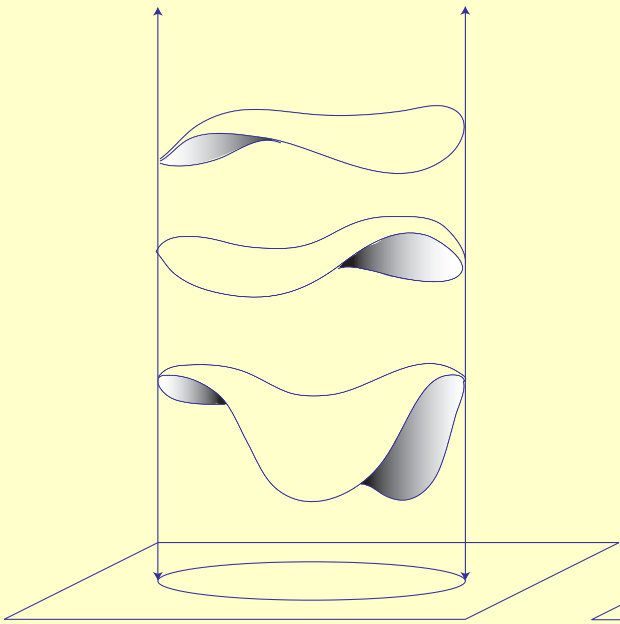
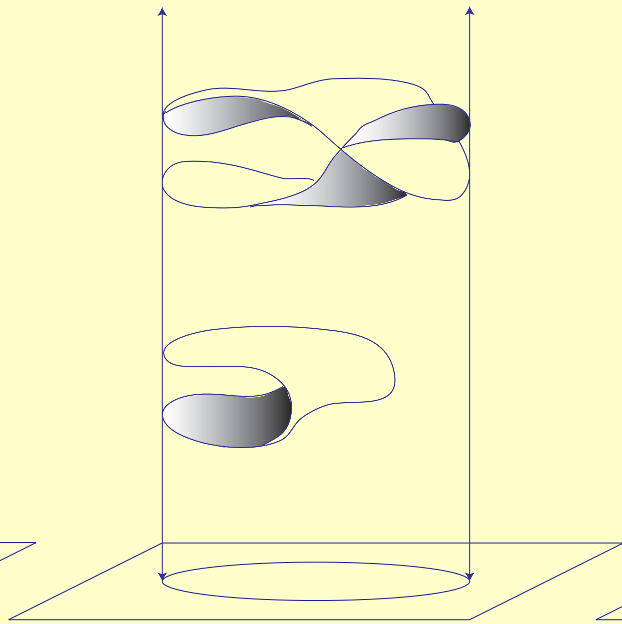


Making natural algebraic decompositions cylindrical

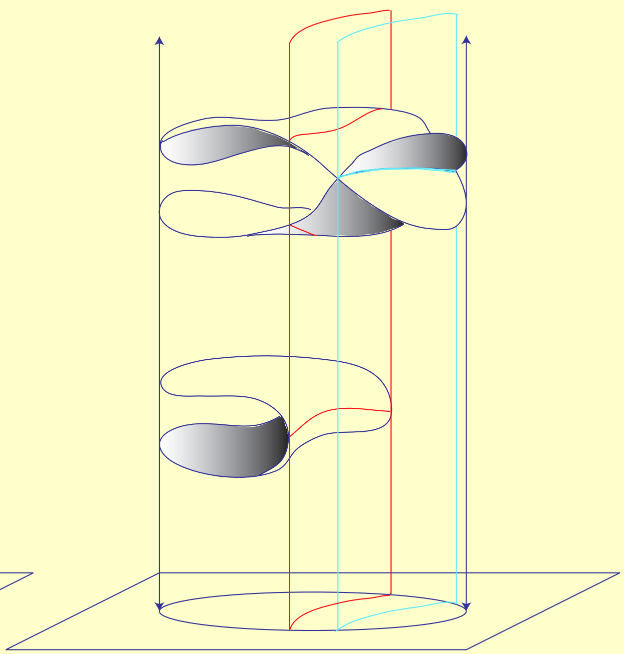
In the pictures below, the "pancakes" are zero sets of polynomials.



cylindrical over region



not cylindrical over region



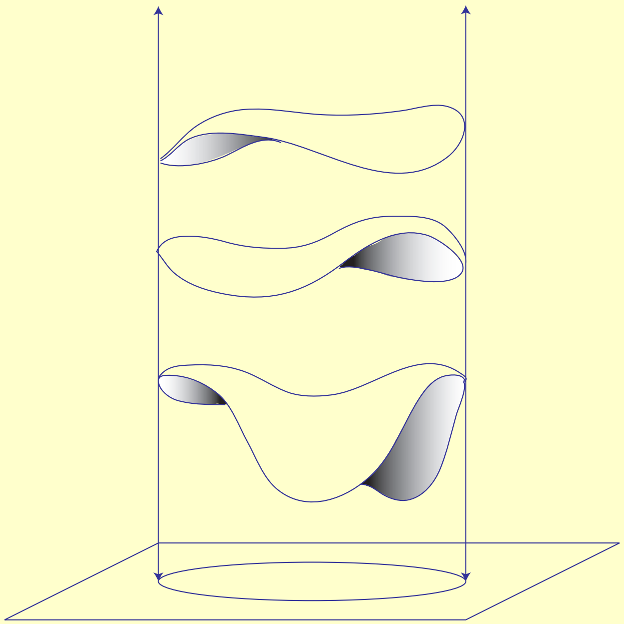
cylindrical over refined regions

Delineability

Let S be a connected subset of $\mathbb{R}^{k-1}$ and let f be a continuous real-valued function on S, and let p be a k-level polynomial that is not nullified (i.e. identically zero) anywhere in S.

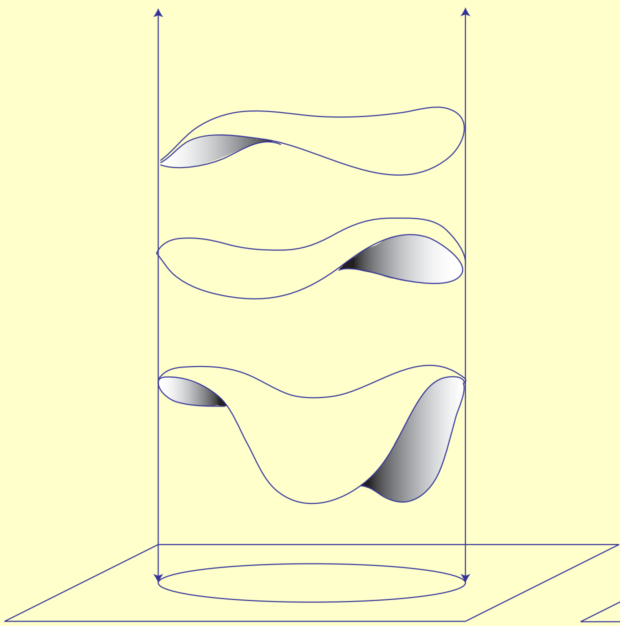
- If $p(\overline{x}, f(\overline{x})) = 0$ for all $\overline{x} \in S$, the graph of f over S is a *section* of p.
- If the zero set of p over S consists of finitely many disjoint sections, p is said to be *delineable* over S.
- A set of k-level polynomials is *delineable* over S if each polynomial is either nullified or delineable over S and if sections of any two elements of the set are either identical or disjoint.

Delineable or Not Delineable!

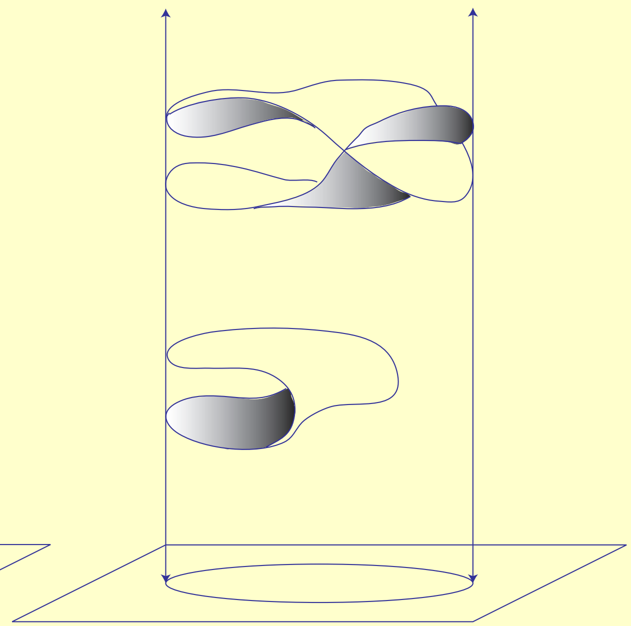


delineable over region

Delineable or Not Delineable!

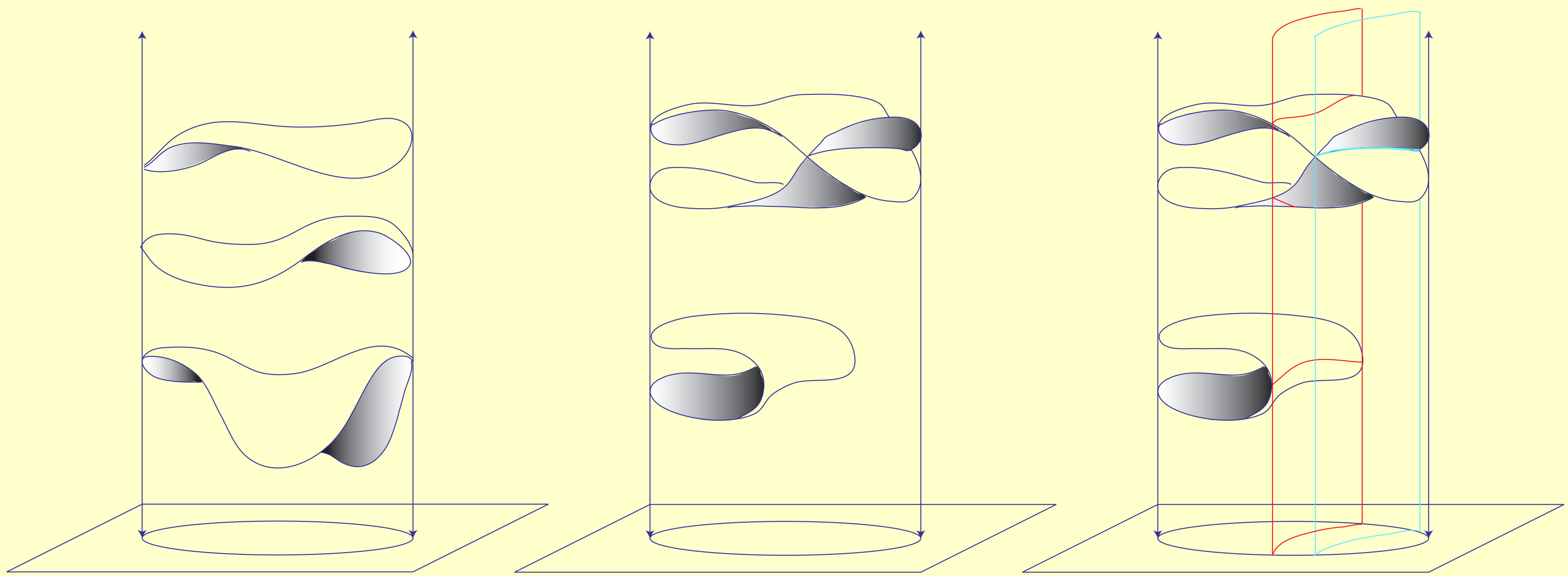


delineable over region



not delineable over region

Delineable or Not Delineable!



delineable over region — not delineable over region — delineable over refined regions

Punchline: If a set of polynomials is *delineable* over region S, the natural algebraic decomposition of $S \times \mathbb{R}$ they define is *cylindrical*.

Delineability and CADs

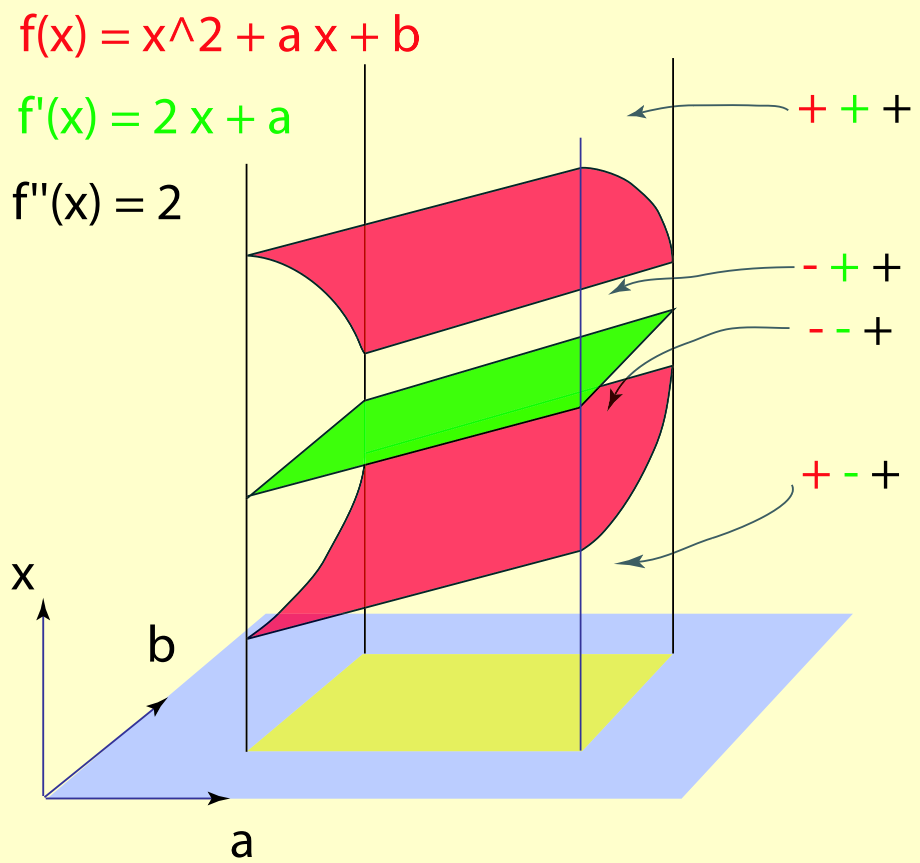
Let $P \subset \mathbb{R}[x_1, \ldots, x_k]$ be a projection factor set.

- If c is a cell in the induced CAD of $(k-1)$-space, the k-level projection factors are delineable over c.

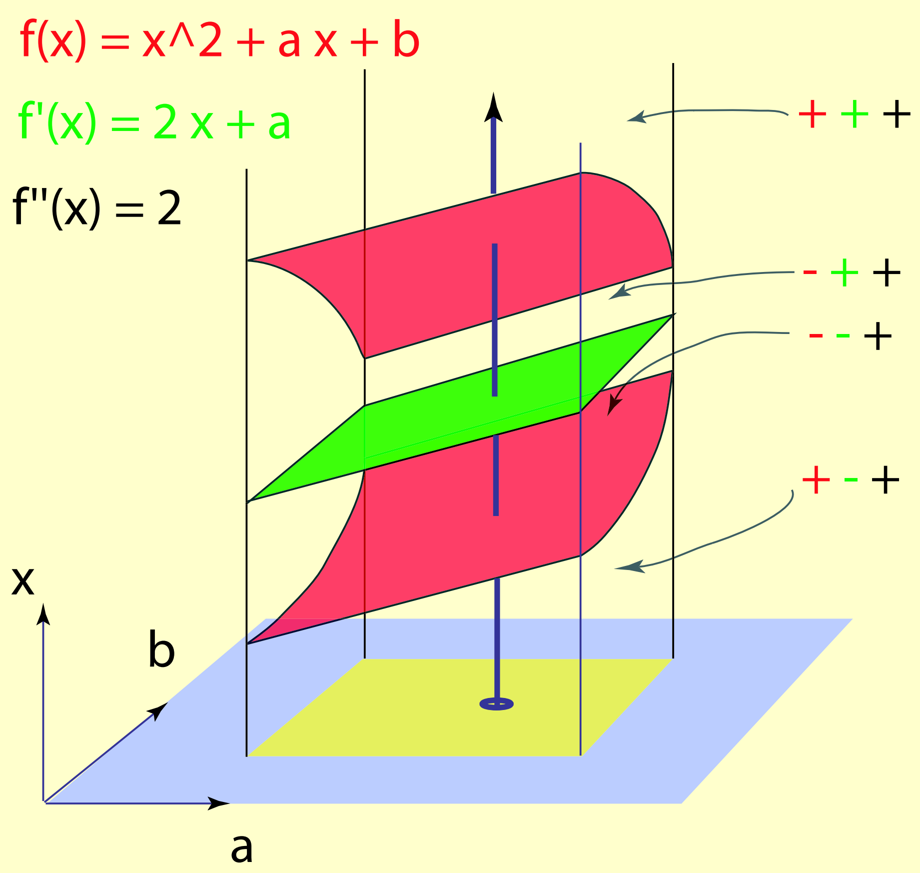
- If A is a set of irreducible $(k+1)$-level polynomials that are delineable over each cell of the CAD defined by P, $A \cup P$ is a projection factor set.

Delineability and the Rolle's Theorem Problem

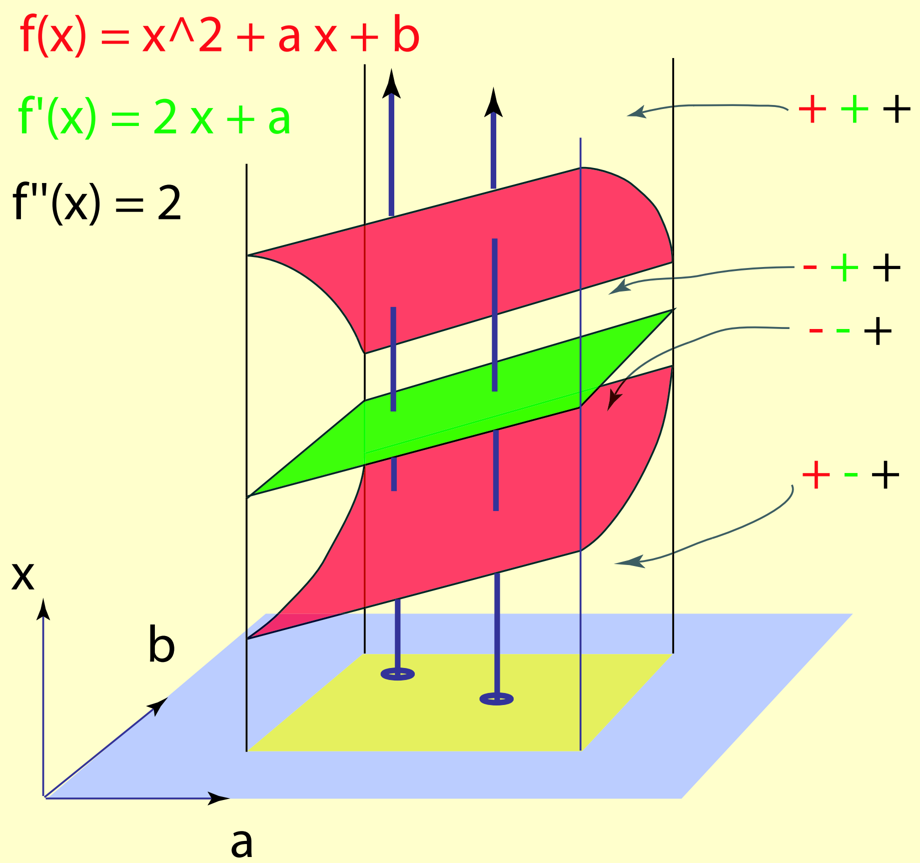
Delineability and the Rolle's Theorem Problem



Delineability and the Rolle's Theorem Problem



Delineability and the Rolle's Theorem Problem



Recall ...

Let $P \subset \mathbb{R}[x_1, \ldots, x_k]$ be a projection factor set.

- If c is a cell in the induced CAD of $(k-1)$-space, the k-level projection factors are delineable over c.

- **If A is a set of irreducible $(k+1)$-level polynomials that are delineable over each cell of the CAD defined by P, $A \cup P$ is a projection factor set.**

Projection Operator

Let A be a set of irreducible polynomials in $x_1, \ldots, x_n$, and let A_k denote the k-level elements of A.

Goal: Construct a projection factor set that contains A.

Define function P such that $P(A_n) \subset \mathbb{R}[x_1, \ldots, x_{n-1}]$ and any projection factor set Q containing the irreducible factors of $P(A_n)$ defines a CAD over whose cells A_n is delineable. I.e. $Q \cup A_n$ is a projection factor set.

The k-level problem "construct a projection factor set containing A" becomes the $(k-1)$-level problem "construct a projection factor set containing $(A - A_k) \cup P(A_k)$".

The function P is called a *projection operator*.

Projection Operator Overview

- There are many projection operators:
  - Collins' projection operator (the original)
  - Hong's projection operator (improves on Collins')
  - McCallum's projection operator
  - Brown-McCallum projection operator (improves McCallum's)
  - "special purpose" projection operators: Collins-McCallum equational constraints, Seidl-Sturm generic CAD, Strzeboński solving strict systems, etc.

- Projection operators that produce small sets are best

Brown-McCallum Projection

- The following is *almost* a projection operator:

$$P(A_k) = \bigcup_{p \in A_k} \{\mathrm{disc}_{x_k}(p), \mathrm{ldcf}_{x_k}(p)\} \cup \bigcup_{p,q \in A_k} \mathrm{res}_{x_k}(p,q)$$

  This is the Brown-McCallum projection.

- The Brown-McCallum projection is smallest, but may fail to produce a CAD such A_k is delineable over each cell. Details about when and why will be left 'til later.

Projection Example 2D

$$A_2 = \{p = 2y^2 - x^2(2x+3),\ q = 2(x+1)y - 1\}$$

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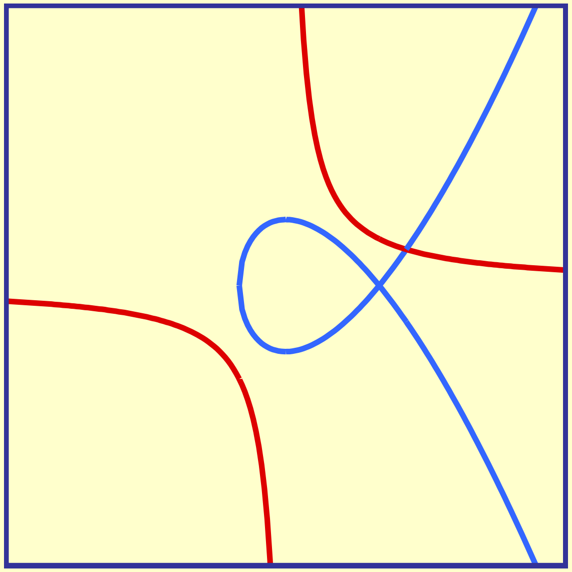

$$P(A_2):$$

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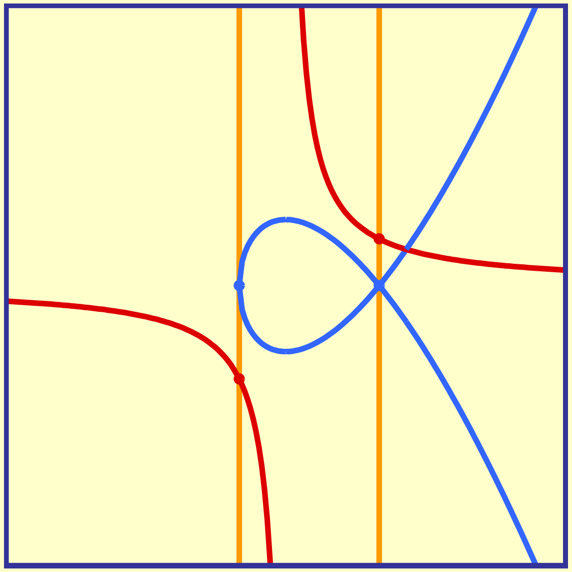

$P(A_2)$:
$\mathrm{disc}_y(p)$

Projection Example 2D

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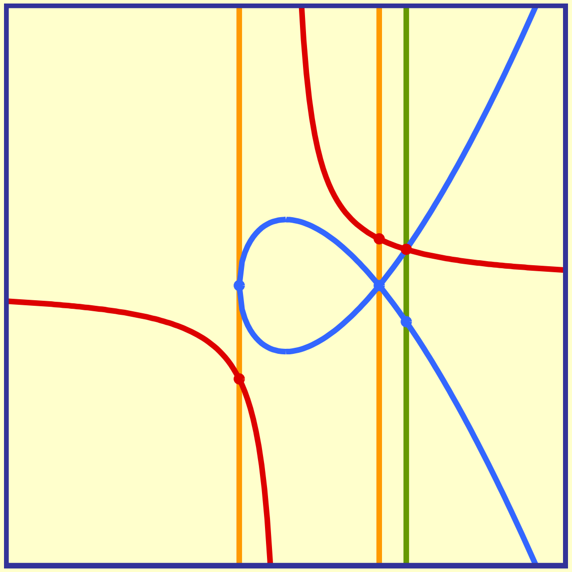

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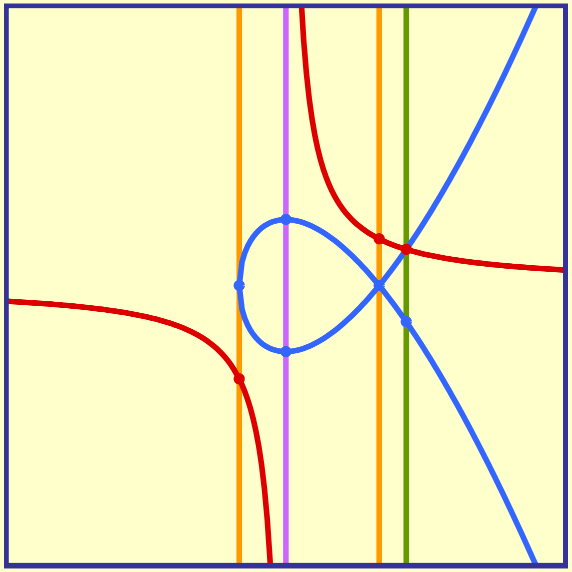

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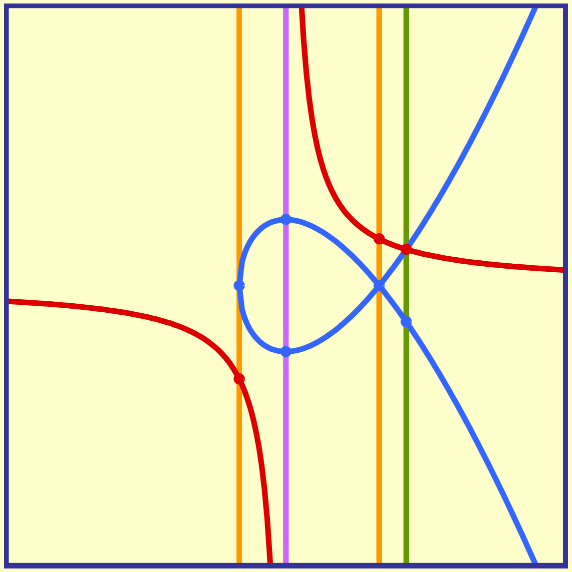

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Projection Example 3D

$$\boxed{\exists z\left[x^2+y^2+z^2-1<0 \wedge 2(x+y)z-1>0\right]}$$

Projection Example 3D

$$\boxed{\exists z \left[x^2 + y^2 + z^2 - 1 < 0 \land 2(x+y)z - 1 > 0\right]}$$

$$A = \{x^2 + y^2 + z^2 - 1, 2(x+y)z - 1, y\}$$

Projection Example 3D

$$\boxed{\exists z \left[x^2 + y^2 + z^2 - 1 < 0 \wedge 2(x+y)z - 1 > 0\right]}$$

$$\begin{aligned} A &= \{x^2 + y^2 + z^2 - 1, 2(x+y)z - 1, y\} \\ P(A) &= \{y^2 + x^2 - 1, y + x, \\ &\quad 4y^4 + 8xy^3 + 8x^2y^2 - 4y^2 + 8x^3y - 8xy + 4x^4 - 4x^2 + 1\} \end{aligned}$$

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$$B = P(A)$$

Projection Example 3D

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P(B) &= \{x + 1, x - 1, x, 32x^6 - 80x^4 + 85x^2 - 32, 2x^2 - 1\}
\end{aligned}$$

Projection Example 3D

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\end{aligned}$$

$$\boxed{P(B)\cup B\cup A \text{ is a projection factor set}}$$

Projection Example 3D

$$\boxed{\exists z\left[x^2+y^2+z^2-1<0 \wedge 2(x+y)z-1>0 \wedge y>0\right]}$$

Projection Example 3D

$$\boxed{\exists z \left[x^2 + y^2 + z^2 - 1 < 0 \wedge 2(x+y)z - 1 > 0 \wedge y > 0\right]}$$

$$A \quad = \quad \{x^2 + y^2 + z^2 - 1, 2(x+y)z - 1, y\}$$

Projection Example 3D

$$\boxed{\exists z \left[x^2 + y^2 + z^2 - 1 < 0 \land 2(x+y)z - 1 > 0 \land y > 0 \right]}$$

$$\begin{aligned} A &= \{x^2 + y^2 + z^2 - 1, 2(x+y)z - 1, y\} \\ A_3 &= \{x^2 + y^2 + z^2 - 1, 2(x+y)z - 1\} \end{aligned}$$

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\\
B &= A \cup P(A_3)
\end{aligned}$$

Projection Example 3D

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\end{aligned}$$

$$\boxed{C = B \cup P(B_2) \text{ is a projection factor set}}$$

Lifting (a.k.a. Stack Construction)

$\boxed{\text{$A$: initial polynomials}} \xrightarrow{\text{Projection}} \boxed{\text{P: projection factor set, $A \subseteq P$}} \xrightarrow{\text{Lifting}} \boxed{\text{D: data structure for the CAD defined by P}}$

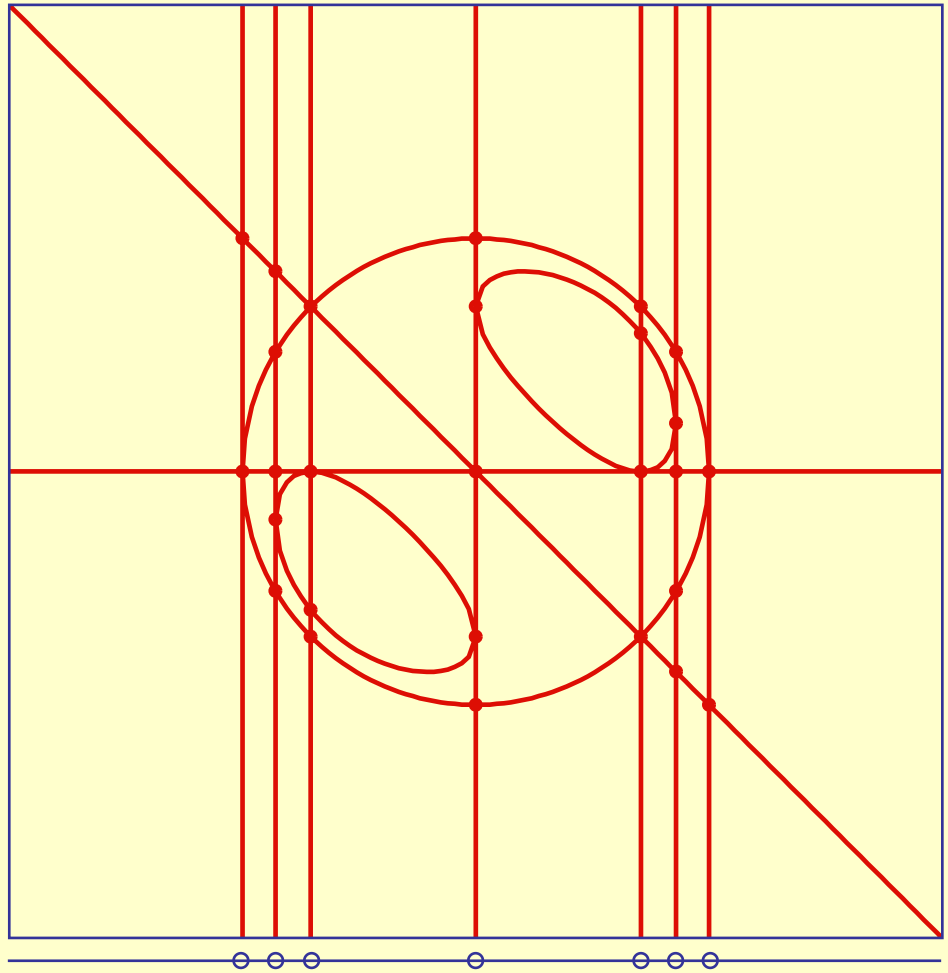
The *lifting* or *stack construction* phase produces an explicit data structure representing the CAD defined implicitly by the projection factor set.

- The data structure represents every cell from the induced CADs of $\mathbb{R}^1, \mathbb{R}^2, \ldots$

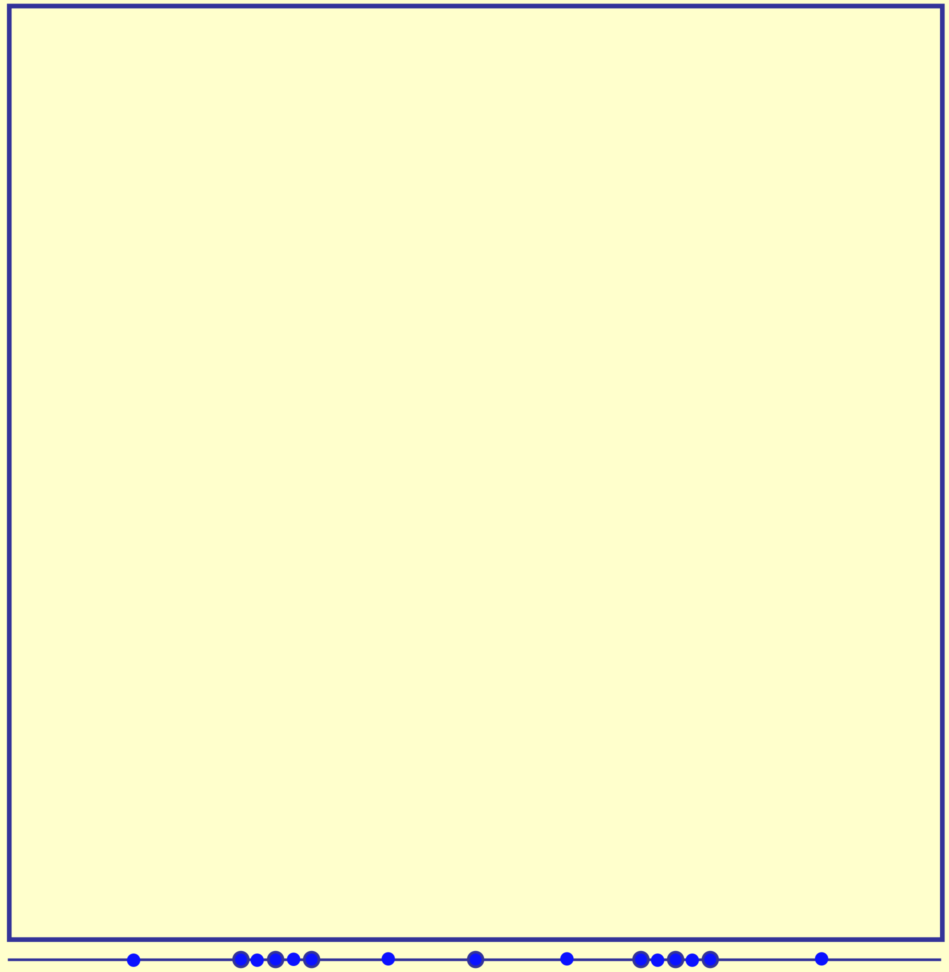
- A cell in the induced CAD of k-space is represented by a sample point from that cell and a list of the cells from the induced CAD of $(k+1)$-space that are stacked over it.

Lifting Example

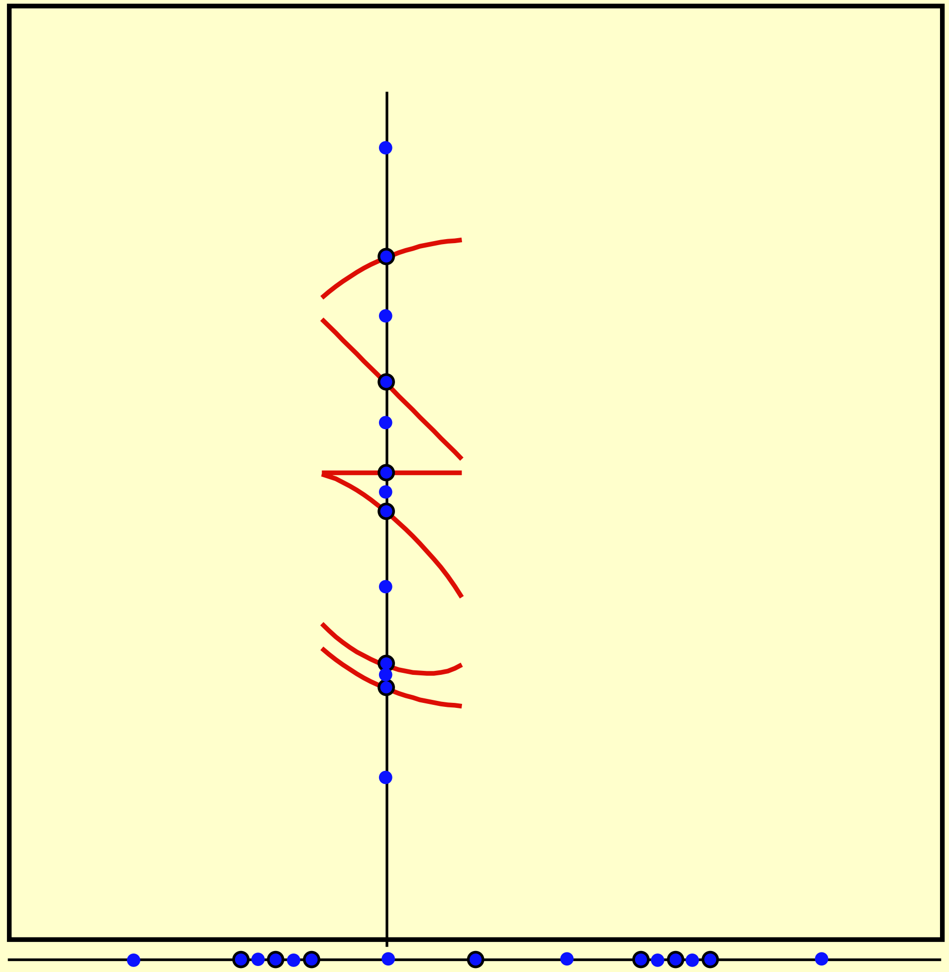
Lifting Example



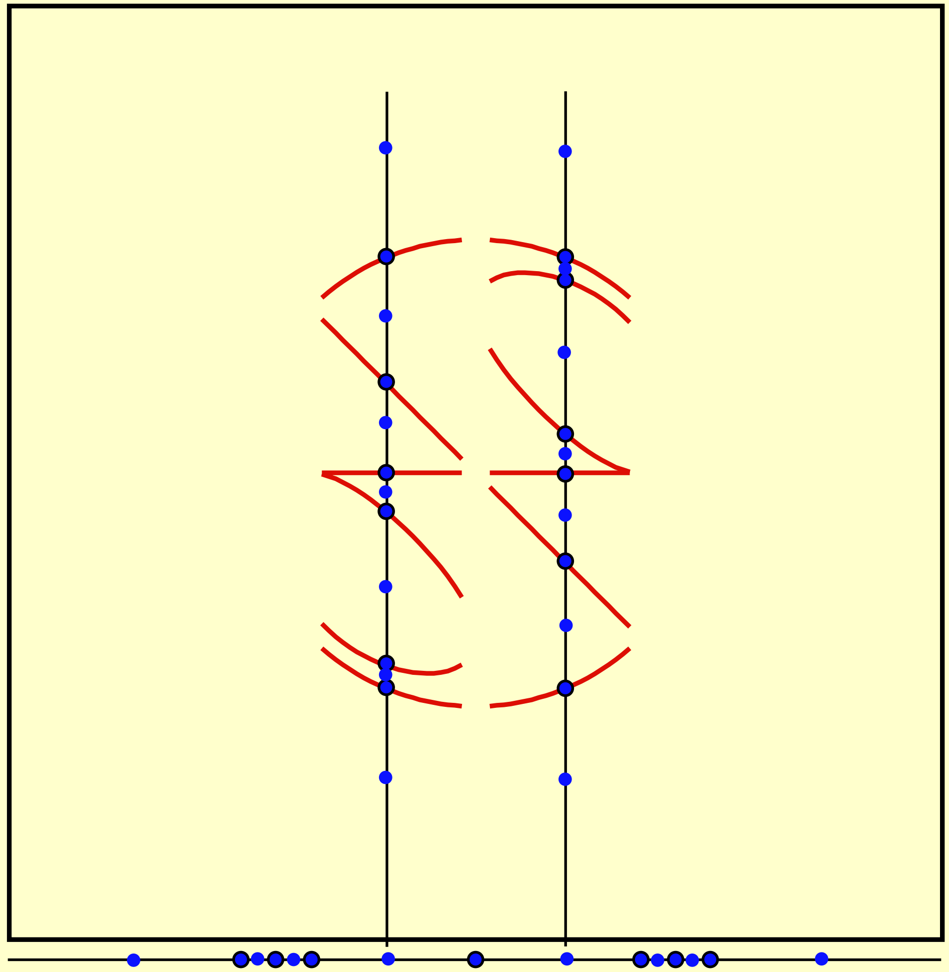
Lifting Example



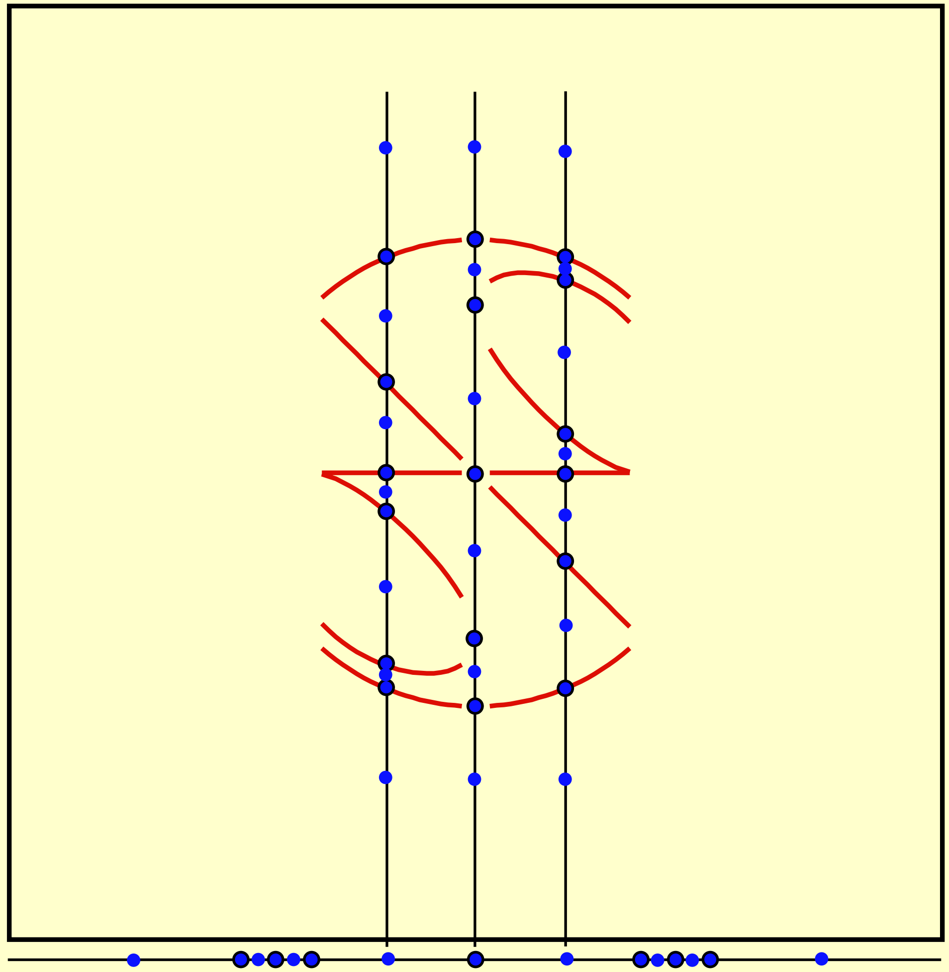
Lifting Example



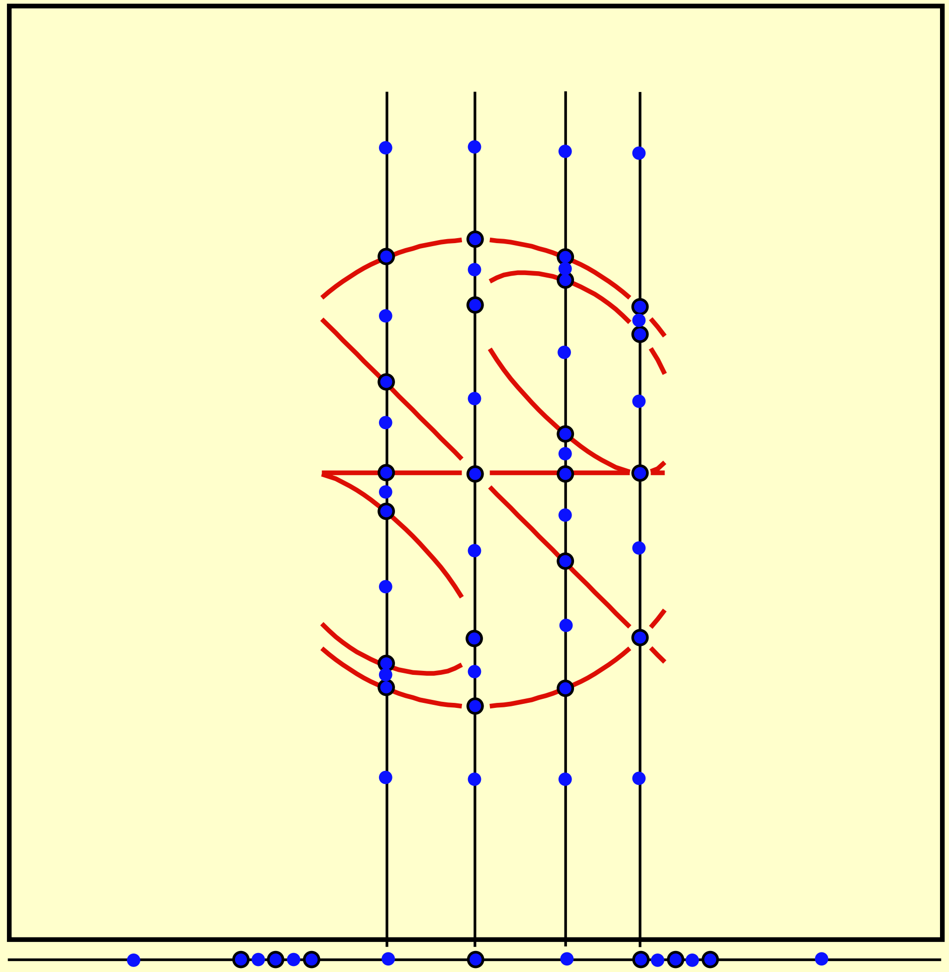
Lifting Example



Lifting Example



Lifting Example



The lifting (stack construction) process

- Let c be a k-level cell with sample point α.
- Let P_{k+1} be the projection factors of level $k+1$.
- To lift over c (i.e. construct the children of c) we
  1. construct $\overline{P}_{k+1} = \{f(\alpha, x_{k+1}) \mid f \in P_{k+1}\}$
  2. Compute $\beta_1 < \cdots < \beta_s$, the roots of elements of $\overline{P}_{k+1}$
  3. Choose rationals $r_1, \ldots, r_{s+1}$ s.t. $r_1 < \beta_1 < r_2 < \cdots < r_s < \beta_s < r_{s+1}$
  4. Set c's children to cells with sample points $(\alpha, r_1), (\alpha, \beta_1), \ldots, (\alpha, r_{s+1})$

Lifting Example Detail

- Let c be the 1-level section cell with sample point $\sqrt{1/2}$

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- Suppose $P_2 = \{y, y^2 + x^2 - 1, y + x, 4y^4 + 8xy^3 + 8x^2y^2 - 4y^2 + 8x^3y - 8xy + 4x^4 - 4x^2 + 1\}$

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- Substituting $x = \sqrt{1/2}$ into P_2 gives $\overline{P}_2 = \{y, y^2 - 1/2, y + \sqrt{1/2}, 2y(2y^3 + 4\sqrt{1/2}y^2 - 2\sqrt{1/2})\}$

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- Roots of $\overline{P}_2$ are $\{-\sqrt{1/2}, 0, \beta, \sqrt{1/2}\}$, where β is the unique root of $x^3 + 2\sqrt{1/2}x^2 - \sqrt{1/2}$ between $1/2$ and $5/8$

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- Children of c are $(\sqrt{1/2}, -1), (\sqrt{1/2}, -\sqrt{1/2}), (\sqrt{1/2}, -1/4), (\sqrt{1/2}, 0), (\sqrt{1/2}, 1/4), (\sqrt{1/2}, \beta), (\sqrt{1/2}, 21/32), (\sqrt{1/2}, \sqrt{1/2}), (\sqrt{1/2}, 1)$

Lifting Issues

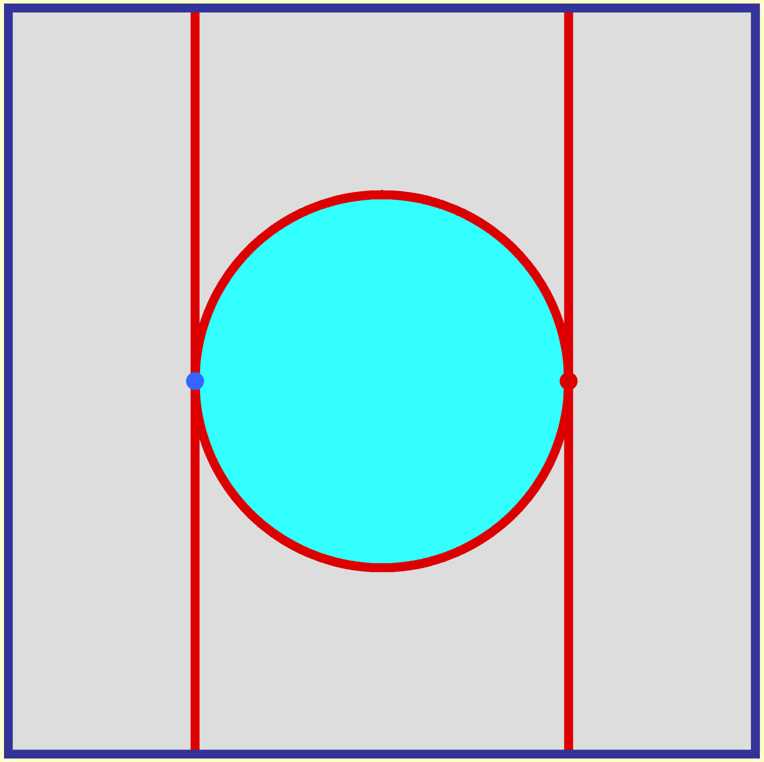
- Typically, more time is spent lifting than doing anything else.
- One must isolate roots of univariate polynomials, often with algebraic number coefficients.
- Algebraic number representation and algorithms are crucial.
- Root isolation algorithm is crucial.
- Use of validated floating-point computation can make a huge difference ... but are a real pain in the kneck to implement!

Solution Formula Construction

Given set S represented by a CAD, construct a Tarski formula defining S.

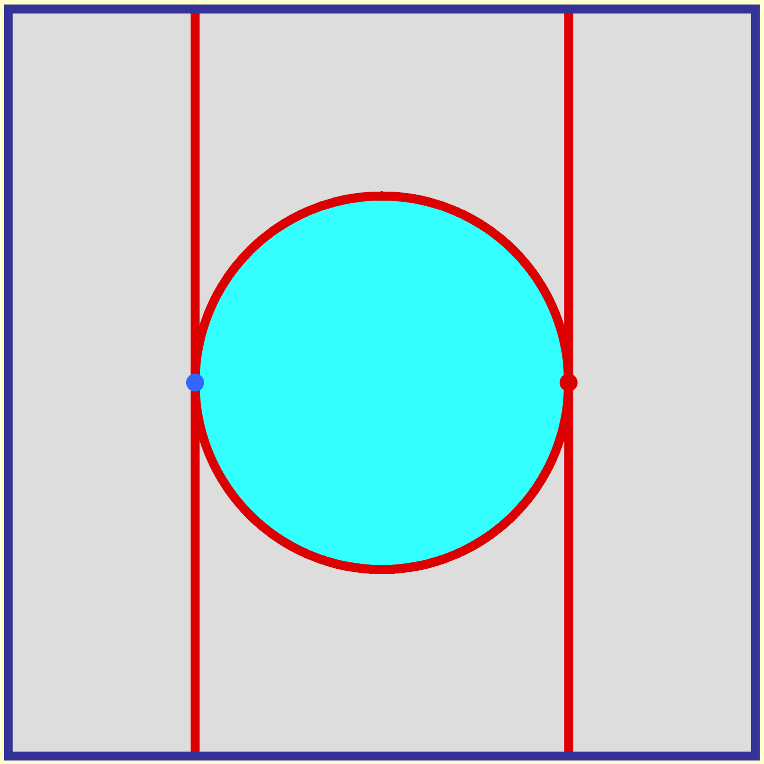
- Construct formula from projection factors (CAD contains complete information about their signs).
- "Simple" formulas are desirable!
- Hong showed how to reduce simple formula construction to a combinatorial optimization problem.
- CAD's ability to provide simple solution formulas is unique.
- Some sets don't have simple defining formulas!
- Note: Allowing the user to state "assumptions" is nice.

Solution Formula Construction Example



$cell$	$P_{1,1}$	$P_{1,2}$	$P_{2,1}$	T/F
$1,1$	$-$	$-$	$+$	F
$2,1$	0	$-$	$+$	F
$2,2$	0	$-$	0	T
$2,3$	0	$-$	$+$	F
$3,1$	$+$	$-$	$+$	F
$3,2$	$+$	$-$	0	F
$3,3$	$+$	$-$	$-$	T
$3,4$	$+$	$-$	0	F
$3,5$	$+$	$-$	$+$	F
$4,1$	$+$	0	$+$	F
$4,2$	$+$	0	0	F
$4,3$	$+$	0	$+$	F
$5,1$	$+$	$+$	$+$	F

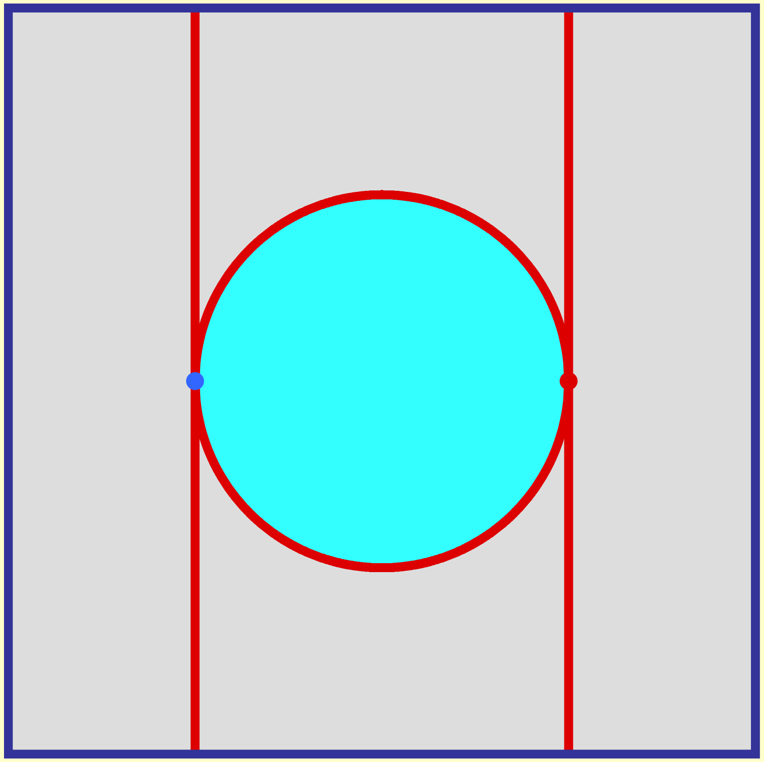
Solution Formula Construction Example



$cell$	$P_{1,1}$	$P_{1,2}$	$P_{2,1}$	T/F
$1,1$	$-$	$-$	$+$	F
$2,1$	0	$-$	$+$	F
$2,2$	0	$-$	0	T
$2,3$	0	$-$	$+$	F
$3,1$	$+$	$-$	$+$	F
$3,2$	$+$	$-$	0	F
$3,3$	$+$	$-$	$-$	T
$3,4$	$+$	$-$	0	F
$3,5$	$+$	$-$	$+$	F
$4,1$	$+$	0	$+$	F
$4,2$	$+$	0	0	F
$4,3$	$+$	0	$+$	F
$5,1$	$+$	$+$	$+$	F

$$P_{2,1} < 0$$

Solution Formula Construction Example

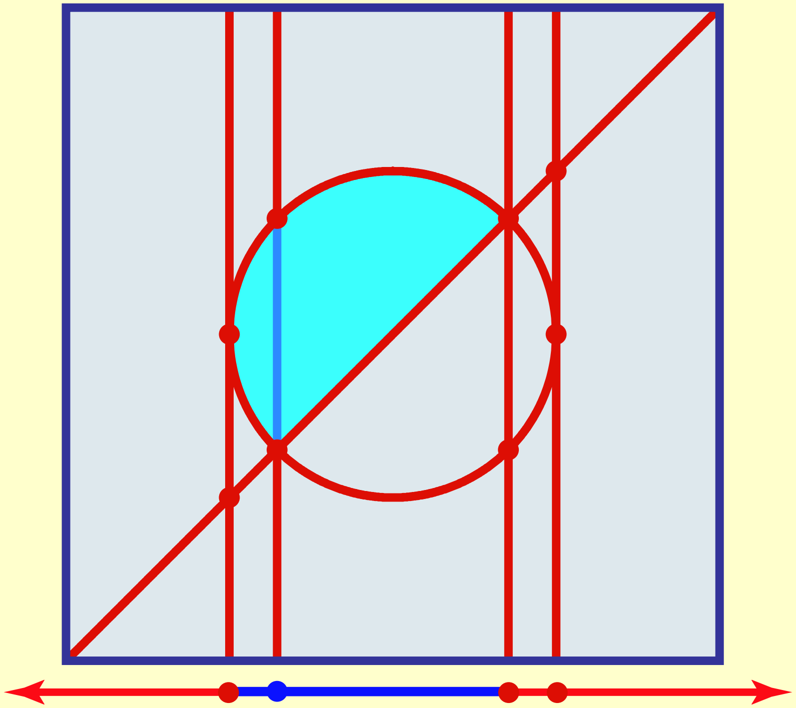


$cell$	$P_{1,1}$	$P_{1,2}$	$P_{2,1}$	T/F
$1,1$	$-$	$-$	$+$	F
$2,1$	0	$-$	$+$	F
$2,2$	0	$-$	0	T
$2,3$	0	$-$	$+$	F
$3,1$	$+$	$-$	$+$	F
$3,2$	$+$	$-$	0	F
$3,3$	$+$	$-$	$-$	T
$3,4$	$+$	$-$	0	F
$3,5$	$+$	$-$	$+$	F
$4,1$	$+$	0	$+$	F
$4,2$	$+$	0	0	F
$4,3$	$+$	0	$+$	F
$5,1$	$+$	$+$	$+$	F

$$P_{2,1} < 0 \;\vee\; P_{1,1} = 0 \wedge P_{2,1} = 0$$

Solution Formula Construction Problem

$$\exists y[x^2 + y^2 - 1 < 0 \ \wedge \ x - y < 0]$$



cell	$x+1$	$x-1$	x^2-2	T/F
1	$-$	$-$	$+$	F
2	0	$-$	$+$	F
3	$+$	$-$	$+$	T
4	$+$	$-$	0	T
5	$+$	$-$	$-$	T
6	$+$	$-$	0	F
7	$+$	$-$	$+$	F
8	$+$	0	$+$	F
9	$+$	$+$	$+$	F

Projection Definability

- Let C be a CAD (with truth values) representing a set S.
- When there is a Tarski formula defining S in which only elements of C's projection factor set appear, C is said to be *projection definable*.
- When a CAD is *not* projection definable, we can
  1. add extra projection factors
  2. extend the language of Tarski formulas
- The projection definability problem tells us that in some sense CADs are more efficient representations of semi-algebraic sets than Tarski formulas.