Case Studies

Epidemiological Example

Recall that the existence of an endemic equalibirium for the SEIT model was equivalent to

$$\exists S \left[\begin{array}{l} -\nu S^2\beta_1^2 + \beta_1\nu S^2\beta_2 + d\beta_1 S r_2 - d^2\beta_2 S + d^2\beta_1 S + \beta_1 S r_1 r_2 - d\nu S\beta_2 \\ +\nu\beta_1 S q r_2 - d\beta_2 r_2 S + d\nu S\beta_1 - \beta_1 S\nu\beta_2 + \beta_1 S r_1 d + \beta_2 d^2 + \nu\beta_2 d \\ +\beta_2 d r_2 = 0 \ \wedge \ 0 < S < 1 \end{array} \right]$$

... under the assumption that all parameters are positive.

$$\nu\beta_1 - dr_2 - d^2 - \nu d > 0 \wedge \nu^2\beta_1^2\beta_2^2 - 2\nu dr_2\beta_1\beta_2^2 - 2\nu d^2\beta_1\beta_2^2 - 2\nu^2 d\beta_1\beta_2^2 + d^2r_2^2\beta_2^2 + 2d^3r_2\beta_2^2$$
$$+ 2\nu d^2r_2\beta_2^2 + d^4\beta_2^2 + 2\nu d^3\beta_2^2 + \nu^2d^2\beta_2^2 - 2\nu r_1r_2\beta_1^2\beta_2 + 2\nu dr_2\beta_1^2\beta_2 - 2\nu^2qr_2\beta_1^2\beta_2 - 2\nu dr_1\beta_1^2\beta_2$$
$$+ 2\nu d^2\beta_1^2\beta_2 + 2\nu^2d\beta_1^2\beta_2 - 2dr_1r_2^2\beta_1\beta_2 - 2d^2r_2^2\beta_1\beta_2 - 2\nu qdr_2^2\beta_1\beta_2 - 4d^2r_1r_2\beta_1\beta_2 - 2\nu dr_1r_2\beta_1\beta_2$$
$$- 4d^3r_2\beta_1\beta_2 - 2\nu qd^2r_2\beta_1\beta_2 - 4\nu d^2r_2\beta_1\beta_2 - 2\nu^2qdr_2\beta_1\beta_2 - 2d^3r_1\beta_1\beta_2 - 2\nu d^2r_1\beta_1\beta_2 - 2d^4\beta_1\beta_2$$
$$- 4\nu d^3\beta_1\beta_2 - 2\nu^2d^2\beta_1\beta_2 + r_1^2r_2^2\beta_1^2 + 2dr_1r_2^2\beta_1^2 + 2\nu qr_1r_2^2\beta_1^2 + d^2r_2^2\beta_1^2 + 2\nu qdr_2^2\beta_1^2 + \nu^2q^2r_2^2\beta_1^2$$
$$+ 2dr_1^2r_2\beta_1^2 + 4d^2r_1r_2\beta_1^2 + 2\nu qdr_1r_2\beta_1^2 + 2\nu dr_1r_2\beta_1^2 + 2d^3r_2\beta_1^2 + 2\nu qd^2r_2\beta_1^2 + 2\nu d^2r_2\beta_1^2 + 2\nu^2qdr_2\beta_1^2$$
$$+ d^2r_1^2\beta_1^2 + 2d^3r_1\beta_1^2 + 2\nu d^2r_1\beta_1^2 + d^4\beta_1^2 + 2\nu d^3\beta_1^2 + \nu^2d^2\beta_1^2 >= 0 \wedge [\nu\beta_1 - r_1r_2 - dr_2 - \nu qr_2 - dr_1 - d^2$$
$$- \nu d > 0 \vee \beta_2 > root_1 \nu^2\beta_1^2\beta_2^2 - 2\nu dr_2\beta_1\beta_2^2 - 2\nu d^2\beta_1\beta_2^2 - 2\nu^2d\beta_1\beta_2^2 + d^2r_2^2\beta_2^2 + 2d^3r_2\beta_2^2 + 2\nu d^2r_2\beta_2^2$$
$$+ d^4\beta_2^2 + 2\nu d^3\beta_2^2 + \nu^2d^2\beta_2^2 - 2\nu r_1r_2\beta_1^2\beta_2 + 2\nu dr_2\beta_1^2\beta_2 - 2\nu^2qr_2\beta_1^2\beta_2 - 2\nu dr_1\beta_1^2\beta_2 + 2\nu d^2\beta_1^2\beta_2$$
$$+ 2\nu^2d\beta_1^2\beta_2 - 2dr_1r_2^2\beta_1\beta_2 - 2d^2r_2^2\beta_1\beta_2 - 2\nu qdr_2^2\beta_1\beta_2 - 4d^2r_1r_2\beta_1\beta_2 - 2\nu dr_1r_2\beta_1\beta_2 - 4d^3r_2\beta_1\beta_2$$
$$- 2\nu qd^2r_2\beta_1\beta_2 - 4\nu d^2r_2\beta_1\beta_2 - 2\nu^2qdr_2\beta_1\beta_2 - 2d^3r_1\beta_1\beta_2 - 2\nu d^2r_1\beta_1\beta_2 - 2d^4\beta_1\beta_2 - 4\nu d^3\beta_1\beta_2$$
$$- 2\nu^2d^2\beta_1\beta_2 + r_1^2r_2^2\beta_1^2 + 2dr_1r_2^2\beta_1^2 + 2\nu qr_1r_2^2\beta_1^2 + d^2r_2^2\beta_1^2 + 2\nu qdr_2^2\beta_1^2 + \nu^2q^2r_2^2\beta_1^2 + 2dr_1^2r_2\beta_1^2$$
$$+ 4d^2r_1r_2\beta_1^2 + 2\nu qdr_1r_2\beta_1^2 + 2\nu dr_1r_2\beta_1^2 + 2d^3r_2\beta_1^2 + 2\nu qd^2r_2\beta_1^2 + 2\nu d^2r_2\beta_1^2 + 2\nu^2qdr_2\beta_1^2 + d^2r_1^2\beta_1^2$$
$$+ 2d^3r_1\beta_1^2 + 2\nu d^2r_1\beta_1^2 + d^4\beta_1^2 + 2\nu d^3\beta_1^2 + \nu^2d^2\beta_1^2]$$

Epidemiological Example

In this model, the condition $\beta_1 > \beta_2$ should also hold, so ...

$$\exists S \left[\begin{array}{l} -\nu S^2\beta_1^2 + \beta_1\nu S^2\beta_2 + d\beta_1 S r_2 - d^2\beta_2 S + d^2\beta_1 S + \beta_1 S r_1 r_2 - d\nu S\beta_2 \\ +\nu\beta_1 S q r_2 - d\beta_2 r_2 S + d\nu S\beta_1 - \beta_1 S\nu\beta_2 + \beta_1 S r_1 d + \beta_2 d^2 + \nu\beta_2 d \\ +\beta_2 d r_2 = 0 \ \wedge \ 0 < S < 1 \end{array} \right]$$

... under the assumption that all parameters are positive, **and** $\beta_1 > \beta_2$. This produces the quantifier-free equivalent formula:

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... under the assumption that all parameters are positive, **and** $\beta_1 > \beta_2$. This produces the quantifier-free equivalent formula:

$$\nu\beta_1 - r_1 r_2 - d r_2 - \nu q r_2 - d r_1 - d^2 - \nu d > 0$$

The Epidemiological Example Shows ...

- the importance of preparing input (recall the substitutions made to eliminate E, I, adn T)
- Order of variables is crucial for this problem (the earlier simple heuristic was used).
- you can only *get* a simple defining formula if the set your Q.E. problem describes *has* a simple defining formula.
- sometimes CADs of high dimension (8 variables for this problem) are feasible.

External Trisectors Problem

Recall: For triangle $\triangle ABC$ the existence of the external trisector of B with respect to A is equivalent to

$$a^2+b^2-c^2 < ab \vee a^2+b^2-c^2 \geq ab \wedge -c\left(a^2+b^2-c^2\right)^3+3a^2b^2c\left(a^2+b^2-c^2\right)$$

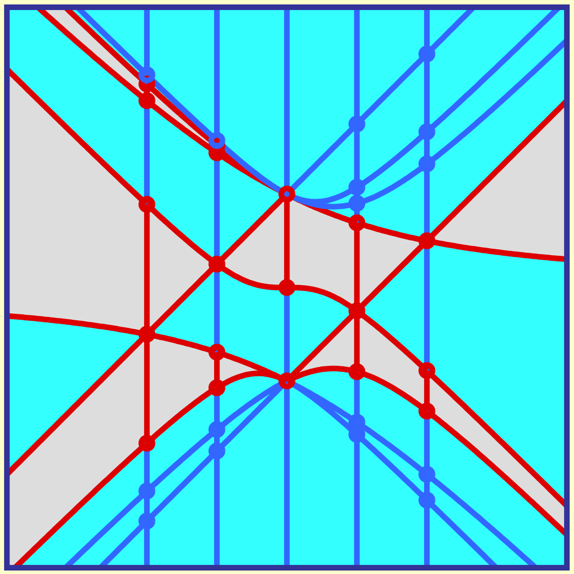
under the assumption that a, b, c define a non-degenerate triangle, i.e. that

$$a > 0 \wedge b > 0 \wedge c > 0 \wedge a < b+c \wedge b < a+c \wedge c < a+b$$

Solution: The above is equivalent to $c^2+bc-a^2 > 0$ under the assumptions.

External Trisectors: a deeper look

Normalize $a = 1$ and view the CAD in the bc-plane:



condition

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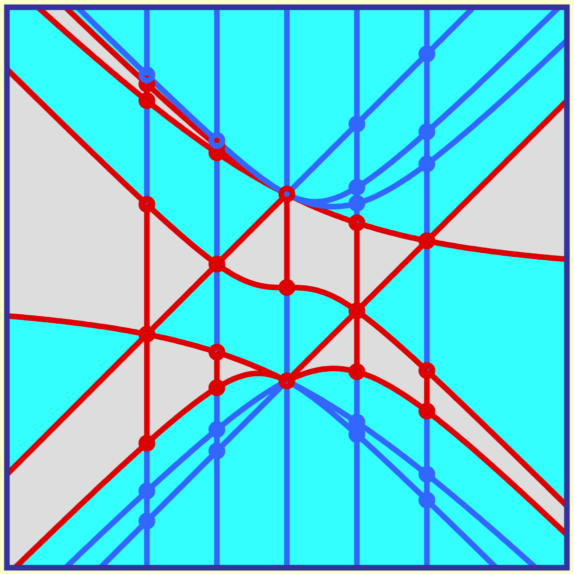

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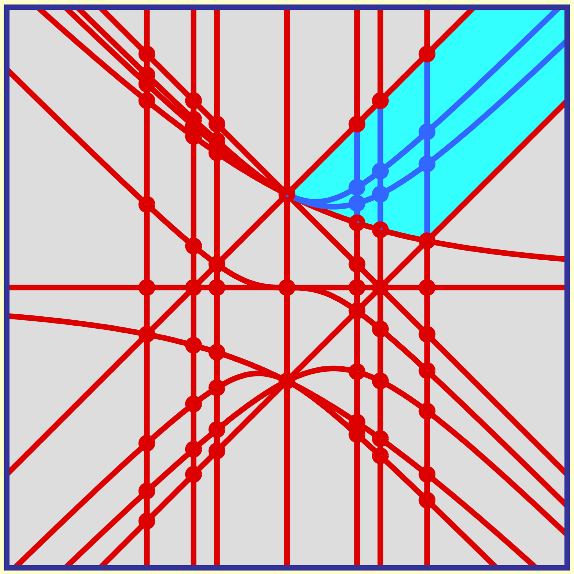

condition with assumptions

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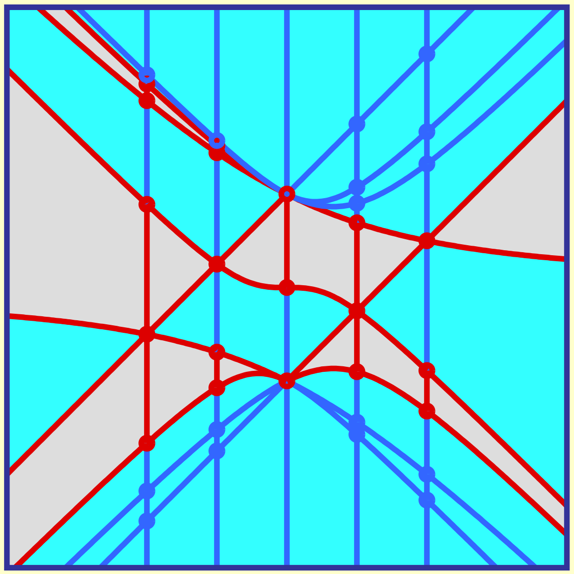
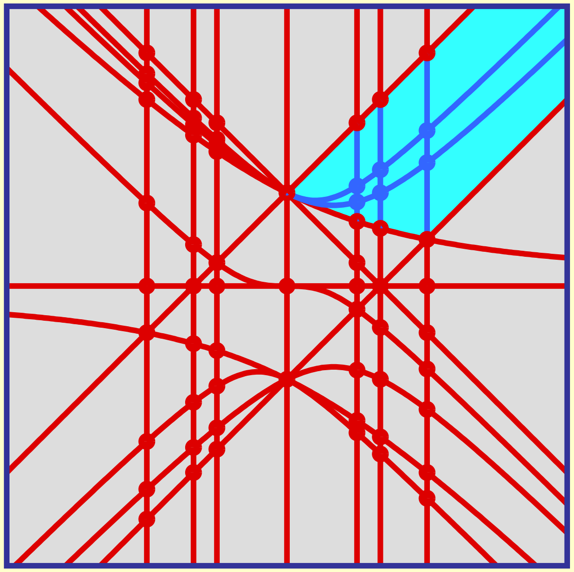
Normalize $a = 1$ and view the CAD in the bc-plane:



condition



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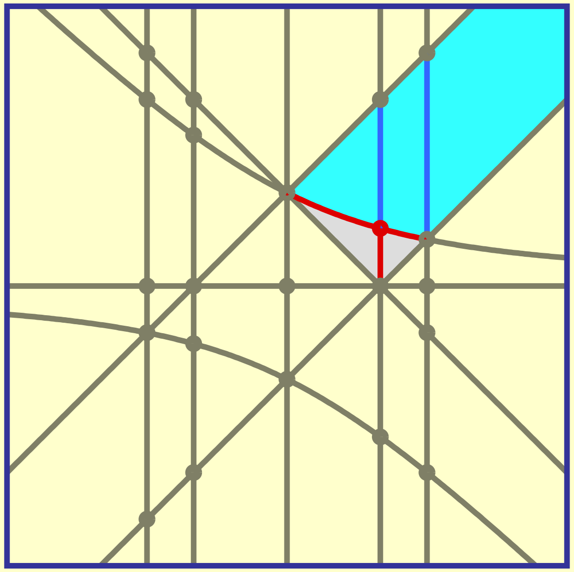

simplified CAD

From this it's easy to read off the formula $c^2 + bc - a^2 > 0$.

Realizable Sign Stack Sequences

- What sign stack sequences are realized by a generic, monic quartic polynomial?

- Consider $f = x^4 + ax^3 + bx^2 + cx + d$.

- Construct CAD for f, f', f'', f''' with order $a \prec b \prec c \prec d \prec x$.

- Consider cell R in induced CAD of $\mathbb{R}^4$.

- Since f, f', f'', f''' are delineable over R, every $(a, b, c, d) \in R$ defines a polynomial with the same sign stack sequence, which can be read off of the stack over R.

- So we can enumerate all realizable sign stack sequences.

Realizable Sign Stack Sequences Cont.

- The problem is only interested in the "generic" situtation, i.e. no common zeros between any pair of f, f', f'', f'''. Thus it suffices to consider only full dimensional cells.

- List sign stack sequence from each full dimensional cell in $\mathbb{R}^4$ (there are 72)

- Remove duplicate sequences (there are now 36, versus 42 legal sequences)

- Compare with list of all legal sign stack sequences to see which legal sequences are not realizable. Here's an example of an unrealizable sequence:

$$\begin{array}{ccccccccccc}
+ & - & - & - & + & + & + & + & - & - & + \\
- & - & + & + & + & - & - & - & - & + & + \\
+ & + & + & - & - & - & - & + & + & + & + \\
- & - & - & - & - & - & + & + & + & + & + \\
+ & + & + & + & + & + & + & + & + & + & +
\end{array}$$

Realizable Sign Stack Sequences: Wrapup

- A trivial (less than 1 second) calculation answers the question: there are unrealizable sign stack sequences for 4th degree polynomials. In fact we can list them all.

- Can easily determine that the thesis' proposition that sequences containing the subsequence $(- + - - +), (+ - - - +), (+ - + + +), (- * + + +)$ are unrealizable still holds if $(- * + + +)$ is dropped.

- Degree 5 case is doable (516 realizable vs. 1000 legal).

- **The Point:** *Make use of the properties of CADs.* Don't view CAD as simply a black box for Q.E. or formula simplification.