1. Given a vector $\mathbf{w} = \langle 2, -1\rangle$, a function $f(x, y) = (3 - x^2)(2 - y^2)$, and a point $P(1, 1)$, answer the following: (a) What is the gradient of f?

Using the product rule, we have

$$\nabla f(x, y) = \langle -2x(2 - y^2), -2y(3 - x^2)\rangle.$$

(b) What is the directional derivative of f at P in the direction of $\mathbf{w}$?

The normalized version of $\mathbf{w}$ is $\dfrac{\mathbf{w}}{|\mathbf{w}|} = \dfrac{1}{\sqrt{5}}\langle 2, -1\rangle$.

Thus,

$$D_{\mathbf{w}}(1, 1) = \nabla f(1, 1) \cdot \frac{\mathbf{w}}{|\mathbf{w}|} = \langle -2, -4\rangle \cdot \frac{1}{\sqrt{5}}\langle 2, -1\rangle = 0.$$

(c) In what direction is f increasing the most at P?

In the direction of the gradient at P which is $\nabla(1, 1) = \langle -2, -4\rangle$.

(d) Let $F(x, y, z) = (3 - x^2)(2 - y^2) - z$ and consider the level surface $F(x, y, z) = 0$. What is the tangent plane at $(1, 1, 2)$? The tangent plane is given by:

$$F_x(1, 1, 2)(x - 1) + F_y(1, 1, 2)(y - 1) + F_z(1, 1, 2)(z - 2) = 0, \text{ or } -2(x - 1) - 4(y - 1) - (z - 2) = 0.$$

(e) Is there a point on the surface $F(x, y, z) = 0$ at which the tangent plane is parallel to the xy-axis? If so, where? If not, why not?

To be parallel to the xy-plane, the tangent plane must have form $z = k$ for some constant k. Thus, we must find a point where $F_x = F_y = 0$. This occurs at five places: $(0, 0, 6)$ and $(\pm\sqrt{3}, \pm\sqrt{2}, 0)$.

2. Professor May B. Wright makes a number of statements. Mark "T" or "F" on the given line depending on whether they are true or false.

(a) F The directional derivative of a given bivariate function, f, and two-dimensional vector $\mathbf{v}$ for a given point (x, y) can be written as $D_{\mathbf{v}}f(x, y) = \nabla f(x) \cdot \mathbf{v}$.

(b) F Any equation with the form $ax^2 + by^2 + cz^2 = d$ describes an ellipsoid for all a, b, and c.

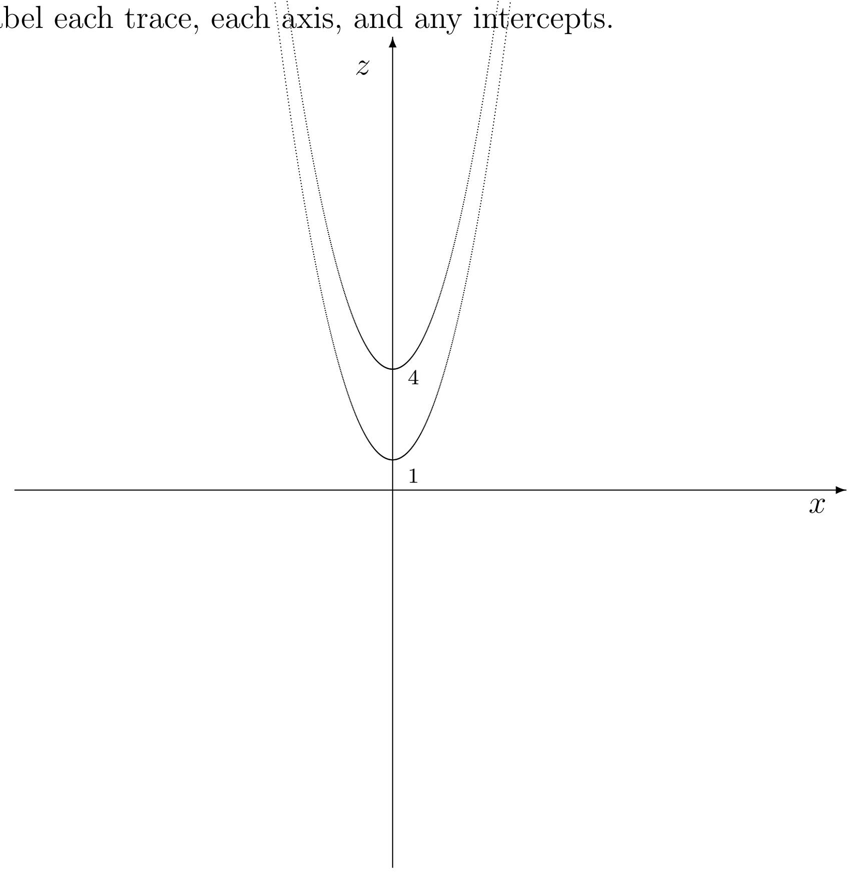
(c) T For all real points, (x, y), there is a unique polar coordinate, (r, θ) so long as we restrict r to be nonnegative and $0 \leq \theta < 2\pi$.

(d) T If $f(x, y) \geq 0$ on a continuous region D, then $\int\int_D f(x, y)dA \geq 0$.

(e) T If $E = \{(x, y, z)|g(y) \leq z \leq h(y), a \leq x \leq b, c \leq y \leq d\}$ and f is integrable on E, then

$$\int\int\int_E f(x, y, z)dV = \int_a^b \int_c^d \int_{g(y)}^{h(y)} f(x, y, z)dzdydx = \int_c^d \int_a^b \int_{g(y)}^{h(y)} f(x, y, z)dzdxdy.$$

3. Sketch traces for $y = 2$ and $y = 1$ for the quadric described by $x^2 + y^2 = z$ using the given axes. Label each trace, each axis, and any intercepts.



The equation describes an elliptic paraboloid

4. Set up and compute the following integrals.

(a) Let $D = \{(x,y)|0 \le x \le 1, \sqrt{x} \le y \le 1\}$ and let $f(x,y) = \cos(y^3)$. Find $\int\int_D f(x,y)dA$.

Here, D is the region from the y-axis to the curve $y = \sqrt{x}$ and bound by the $y = 1$ line. The key is to switch the bounds given to obtain

$$\int_0^1 \int_0^{y^2} \cos(y^3)dxdy = \int_0^1 y^2\cos(y^3)dy = \frac{1}{3}\int_0^1 \cos(u)du = \frac{1}{3}\sin(1)$$

(b) Using polar coordinates, let $D = \{(r,\theta)|0 \le r \le \frac{1}{\cos(\theta)}, -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}\}$ and let $f(r,\theta) = e^{r\sin(\theta)}$. *Hint: convert to Cartesian coordinates.*

Note that the curve $r = \frac{1}{\cos(\theta)}$ corresponds, in Cartesian coordinates to the line $x = 1$. Also, the bounds $-\pi/4 \le \theta \le \pi/4$ correspond to the lines $y = x$ and $y = -x$. Thus, $D = \{(x,y)|0 \le x \le 1, -x \le y \le x\}$ and

$$\int\int_D f(x)dA = \int_0^1 \int_{-x}^{x} e^y dydx = \int_0^1 (e^x + e^{-x})dx = e + \frac{1}{e} - 2.$$

5. Set up the interated integrals that will calculate the volume of the following regions, but **do not solve!** (a) Let E denote the cylinder that passes through a unit circle centered at the origin and is parallel to the y-axis intersected with the cube: $\{(x,y,z)|0 \leq x \leq 4, 0 \leq y \leq 4, 0 \leq z \leq 4\}$.

The key is to remember that the xy projection is just a rectangle.

The volume is:

$$\int\int\int_E dV = \int_0^4 \int_0^1 \int_0^{\sqrt{1-x^2}} dzdxdy.$$

(b) Let E denote the region bounded by the quadric surface defined by $x^2 - y^2 - z^2 = 1$ and a sphere of radius 2 centered at the origin. *The problem statement should have also said: Additionally, E is bounded so that x, y, and z are all nonnegative.*

Two keys:

- Recognize that two integrals are needed as the z bounds change.
- Find the intersection of the projection in the xy-plane (when $z = 0$) by solving $x^2 - y^2 = 1$ and $x^2 + y^2 = 4$ to obtain $(\sqrt{5/2}, \pm\sqrt{3/2})$.

The volume is:

$$\int\int\int_E dV = \int_1^{\sqrt{\frac{5}{2}}} \int_{-\sqrt{x^2-1}}^{\sqrt{x^2-1}} \int_{-\sqrt{x^2-y^2-1}}^{\sqrt{x^2-y^2-1}} dzdydx + \int_{\sqrt{\frac{5}{2}}}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dzdydx.$$

SM221, Exam 2 November 4, 2011

Name (please print): ____________________

Only write your name above!

Instructions:

- No books, notes, or calculators are allowed.
- Show all work clearly. (little or no credit will be given for a numerical answer without the correct accompanying work. Partial credit is given where appropriate.)
- If you need more space than is provided, use the back of the previous page.
- Please read the question carefully. If you are not sure what a question is asking, ask for clarification.
- If you start over on a problem, please CLEARLY indicate what your final answer is, along with its accompanying work.

Problem	Points	Score
1	1	
2	1	
3	1	
4	1	
5	1	
Total	5	