Basic feasible solutions: A basic solution which is nonnegative.

Basic solution: For a canonical form linear program (see below), a basic solution is a vector $\mathbf{x}$ where, for a given basis, $\mathcal{B}$, if $j \notin \mathcal{B}$ then $x_j = 0$. The variables x_j for $j \in \mathcal{B}$ solve the square linear system $B\mathbf{x}(\mathcal{B}) = \mathbf{b}$. Thus, because B is invertible,

$$\mathbf{x}(\mathcal{B}) = B^{-1}\mathbf{b}.$$

Basic variable: For a basic solution, $\mathbf{x}$, with basis $\mathcal{B}$, any variable x_j where $j \in \mathcal{B}$.

Basis: For a canonical form linear program (see below), a basis is a set, $\mathcal{B}$, of indices corresponding to m linearly independent columns. For an n dimensional vector $\mathbf{x}$, we write $\mathbf{x}(\mathcal{B})$ to denote a vector with components x_j for $j \in \mathcal{B}$. For the matrix A, we write $A(\mathcal{B})$ to denote the submatrix from taking the m columns of A corresponding to $\mathcal{B}$.

Bland's rule: An anti-cycling pivot rule where the entering variable is the nonbasic variable with a positive reduced cost and the smallest index. If the min. ratio test results in a tie, then the leaving variable is the basic variable chosen is the one with the smallest index.

Blending constraint: For a product made from a "blend" of different items, a constraint such that the percentage of one or more of the items (or even a characteristic of an item) is constrained. To linearize such a constraint, the denominator is typically multiplied through (which assumes that at least a positive amount of the product is made). For example, suppose that a gasoline is made by blending two types of crude oil, type A and type B. Let x_A denote the amount of oil A used and x_B the amount of oil B used. If A has an octane rating of 93 and B an octane rating of 88, then to keep the average octane rating of the gas at least 91, the **nonlinear** constraint

$$\frac{93x_A + 88x_B}{x_A + x_B} \geq 91$$

could be used. To make this constraint **linear** and therefore valid for a linear program, use:

$$93x_A + 88x_B \geq 91(x_A + x_B).$$

Canonical form linear program: A linear program in the form

$$\begin{array}{ll} \text{max or min} & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0, \end{array}$$

where A is an $m \times n$ real matrix with full-row rank, $\mathbf{c}$ is an n dimensional real vector, and $\mathbf{b}$ is an m dimensional vector. The decision variables are the n-dimensional vector $\mathbf{x}$. Note that the objective can be minimization or maximization.

Combinatorial Optimization Problem: A mathematical program where the variables are restricted to zero or one.

Continuous Variable: A decision variable that can take non-integer values.

Dantzig's rule: A pivot rule where the entering (nonbasic) variable is the one with the largest reduced cost (for a maximization problem).

Decision Variable: A variable in a mathematical program that can be changed.

Decision Vector: A vector of some or all (usually all) of the decision variables in a mathematical program.

Degenerate basic feasible solution: A basic feasible solution where one or more of the basic variables is zero.

Discrete Variable: A decision variable that can only take integer values.

Feasible Solution: A solution that satisfies all the constraints.

Feasible Region: The set of all feasible solutions, i.e., $\mathcal{S}$.

Graph: A set, $\mathcal{V}$, of *nodes* (or vertices) and a set of pairs of nodes, called *edges* (or arcs), $\mathcal{E}$, typically written $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and can be drawn as (arbitrarily placed) small circles in space for each $i \in \mathcal{V}$ with a line between nodes i and j if $(i,j) \in \mathcal{E}$. We typically use *directed* graphs, which means the order of the edge matters, i.e., $(i,j) \neq (j,i)$ for $i \neq j$. Note that we also assume that graphs do not have *loops*, i.e., edges from a node to itself, or *multiedges*, multiple copies of an edge.

Infeasible Problem: A mathematical program where $\mathcal{S} = \emptyset$, i.e., where no feasible solution exists.

Infeasible Solution: A solution, $\mathbf{x}$, that does not satisfy the constraints, i.e., $\mathbf{x} \notin \mathcal{S}$.

Linear Program (LP): A mathematical program where f is linear and $\mathcal{S}$ is determined by linear inequalities and equalities. This can be written as

$$\begin{array}{lll} \max & c_1 x_1 + \ldots + c_n x_n & \\ \text{s.t.} & a_{i1} x_1 + \ldots + a_{in} x_n & \leq b_i, i = 1, \ldots, m. \end{array}$$

We can show that **every** LP can be written in this form. Linear programs assume *additivity, certainty, divisbility,* and *proportionality.*

Mathematical Program: Also called an optimization problem, a mathematical program consists of a given set $\mathcal{S} \subset \mathbb{R}^n$ and function $f : \mathcal{S} \to \mathbb{R}$. Then, the mathematical program is

$$\max\{f(\mathbf{x}) : \mathbf{x} \in \mathcal{S}\} \text{ or } \min\{f(\mathbf{x}) : \mathbf{x} \in \mathcal{S}\}.$$

Minimum ratio test: For a canonical form linear program, basic solution, $\mathbf{x}$, with basis, $\mathcal{B}$, and simplex direction $\mathbf{d}^{(j)}$, the minimum ratio test is the calculation used to determine the maximum step size, λ, that can be used before $\mathbf{x}+\lambda\mathbf{d}^{(j)}$ is not feasible. Because $\mathbf{x}+\lambda\mathbf{d}^{(j)}$ satisfies equality constraints for any λ, the step size is determined in order to satisfy the nonnegativity constraints, i.e., λ is the maximum value such that, for all i,

$$x_i + \lambda d_i^{(j)} \geq 0.$$

For nonbasic indices i, $x_j + \lambda d_i^{(j)} \geq 0$ so the minimum ratio test is calculated via

$$\lambda = \min\{\frac{x_i}{-d_i^{(j)}} | i \in \mathcal{B}, d_i < 0\}.$$

Nonbasic variable: For a basic solution, $\mathbf{x}$, with basis $\mathcal{B}$, any variable x_j where $j \notin \mathcal{B}$. All nonbasic variables are zero, i.e., $x_j = 0$ for $j \notin \mathcal{B}$.

Pivot rule: How an entering is picked in Simplex. Pivot rules are only used if there is an improving direction (i.e., at least one positive reduced cost for a max problem or at least one negative reduced cost for a min problem). Some pivot rules also specify how leaving variables are chosen in the event of a tie in the minimum ratio test.

Optimal Solution: A feasible solution, $\mathbf{x}^*$, that has, for all $\mathbf{x} \in \mathcal{S}$:

$$f(\mathbf{x}^*) \geq f(\mathbf{x}) \text{ for a maximization problem, or;}$$

$$f(\mathbf{x}^*) \leq f(\mathbf{x}) \text{ for a minimization problem.}$$

Reduced cost: For a canonical form linear program and a basic feasible solution with basis, $\mathcal{B}$, and basis matrix $B = A(\mathcal{B})$, the reduced cost of the nonbasic variable with index $j \notin \mathcal{B}$ is the directional derivative of the objective function, $\mathbf{c}^\top \mathbf{x}$ in the simplex direction $\mathbf{d}^{(j)}$, i.e.,

$$\bar{c}_j = D_{\mathbf{d}^{(j)}}(\mathbf{c}^\top \mathbf{x}) = \nabla(\mathbf{c}^\top \mathbf{x}) \cdot \mathbf{d}^{(j)} = \mathbf{c} \cdot \mathbf{d}^{(j)} = c_j - c(\mathcal{B})^\top B^{-1} A_j.$$

Reduced cost optimality conditions: For a canonical form linear program, a basic feasible solution with basis, $\mathcal{B}$, is optimal for a maximization problem if, for all $j \notin \mathcal{B}$,

$$\bar{c}_j \leq 0.$$

For a minimization problem, the condition is that for all $j \notin \mathcal{B}$,

$$\bar{c}_j \geq 0.$$

Simplex direction: For a canonical form linear program and a basic feasible solution, $\mathbf{x}$, with basis, $\mathcal{B}$, and basis matrix $B = A(\mathcal{B})$, the simplex direction of the nonbasic variable with index $j \notin \mathcal{B}$ is an n-dimensional vector $\mathbf{d}^{(j)}$ where $d_j^{(j)} = 1$, for $i \notin \mathcal{B}, i \neq j$, $d_i^{(j)} = 0$ and

$$\mathbf{d}^{(j)}(\mathcal{B}) = -B^{-1}A_j.$$

Note that A_j is the jth column of A. Also recall that the derivation of $\mathbf{d}^{(j)}$ was based on (1) changing only one non-basic variables, and (2) satisfying all the equality constraints for any change from the basic feasible solution, $\mathbf{x}$, i.e., for any λ and $j \notin \mathcal{B}$.

$$A(\mathbf{x} + \lambda \mathbf{d}^{(j)}) = \mathbf{b}.$$

Solution: A particular setting of the decision vector. Note that the solution may not satisfy all the constraints.

Unbounded Mathematical Program: A mathematical program where for any $K > 0$, for a maximization, there exists a feasible solution $\mathbf{x}$ with $f(\mathbf{x}) > K$. For a minimization problem, for any $K < 0$ there is a feasible solution $\mathbf{x}$ with $f(\mathbf{x}) < K$.