Generic Minimum Cost Flow formulation
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We assume $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a given directed graph. The linear programming formulation for the **Minimum Cost Flow** problem is as follows.

Index sets:
Both $\mathcal{V}$, the set of nodes, and $\mathcal{E}$, the set of edges, are index sets. Note that there is a constraint for every node.

Parameters:
There are edge cost per unit flow, c_{ij}, and capacities, u_{ij}, for every edge $(i,j) \in \mathcal{E}$, and node supply/demand, b_i, for every node $i \in \mathcal{V}$.

Decision variables:
The decision variable is x_{ij}, for every edge $(i,j) \in \mathcal{E}$, which is the flow from node i to node j.

Formulation

$$
\begin{array}{llll}
\min & \sum\limits_{(i,j)\in\mathcal{E}} c_{ij}x_{ij} & & (a) \\
\text{s.t.} & \sum\limits_{j:(i,j)\in\mathcal{E}} x_{ij} - \sum\limits_{j:(j,i)\in\mathcal{E}} x_{ji} = b_i, & \text{for all } i \in \mathcal{V} & (b) \\
& 0 \le x_{ij} \le u_{ij} & \text{for all } (i,j) \in \mathcal{E} & (c)
\end{array}
$$

Discussion:
The objective, (a), minimizes the sum of edge flows multiplied by edge costs. Constraints (b) are called *flow-balance constraints*, and have the sum of flow out of a node i minus the sum of flow into the node i equal to the supply/demand at node i. Note that the node i is FIXED for each of these constraints and the red font is to draw attention to this fact. The nonnegativity and edge capacity is enforced in constraints (c).

Variations:
The **shortest path problem** has no capacities (i.e., $u_{ij} = \infty$) for all $(i,j) \in \mathcal{E}$) and two designated nodes $s, t \in \mathcal{V}$ such that for all $i \in \mathcal{V}$:

$$
b_i = \begin{cases} 1 & i = s, \\ -1 & i = t, \\ 0 & \text{otherwise.} \end{cases}
$$

The **maximum flow problem** also has two designated nodes s and t as well as an arc (t,s) in the graph. However, there are capacities, but the costs are defined for each edge (i,j) as:

$$
c_{ij} = \begin{cases} -1 & (i,j) = (t,s) \\ 0 & \text{otherwise.} \end{cases}
$$