Integral example for 11/16
SM221, Fall 2011
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Let $F(x,y,z) = \langle x^3\sqrt{y}, x\rangle$ and $C$ be the curve defined by $r(t) = \langle t, 3t^2+1\rangle$ for $0 \le t \le 1$. Note that we could write the line integral as

$$\int_C F \cdot dr = \int_C x^3\sqrt{y}\, dx + x\, dy.$$

Also, as shown in class

$$\begin{aligned}
\int_C F \cdot\, dr &= \int_0^1 F(r(t)) \cdot r'(t)\, dt \\
&= \int_0^1 \langle t^3\sqrt{3t^2+1}, t\rangle \cdot \langle 1, 6t\rangle\, dt \\
&= \int_0^1 (t^3\sqrt{3t^2+1} + 6t^2)\, dt \\
&= \int_0^1 t^3\sqrt{3t^2+1}\, dt + 6\int_0^1 t^2\, dt.
\end{aligned}$$

For the first integral, the substitution, $u = 3t^2+1$ with $du = 6t\, dt$ results in

$$\begin{aligned}
\int_C F \cdot dr &= \int_0^1 t^3\sqrt{3t^2+1}\, dt + 6\int_0^1 t^2\, dt \\
&= \int_0^1 \tfrac{1}{3}\big((3t^2+1)-1\big)\sqrt{3t^2+1}\tfrac{1}{6}(6t\, dt) + 6\int_0^1 t^2\, dt \\
&= \tfrac{1}{18}\int_1^4 (u-1)\sqrt{u}\, du + 2t^3|_0^1 \\
&= \tfrac{1}{18}\int_1^4 (u^{3/2} - u^{1/2})\, du + 2 \\
&= \tfrac{1}{18}(\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2})|_1^4 + 2 \\
&= \tfrac{1}{18}\left(\big(\tfrac{2}{5}(32) - \tfrac{2}{3}(8)\big) - \big(\tfrac{2}{5}(1) - \tfrac{2}{3}(1)\big)\right) + 2 \\
&= \tfrac{328}{135}.
\end{aligned}$$