Review Notes: Exam 3

1 Three-Dimensional Coordinates

A. Distance Formula between points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

B. Equation of sphere with radius r and center (h, k, l)

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.$$

C. Example of transforming to an equation of a sphere:

$$x^2 - 4x + y^2 + 16y + z^2 - 6z = 9$$

Divide the coefficients of x, y and z **with the sign** and add the results to both sides of the equation

$$x^2 - 4x + (-2)^2 + y^2 + 16y + (+8)^2 + z^2 - 6z + (-3)^2 = 9 + (-2)^2 + (+8)^2 + (-3)^2$$

You have just added the perfect numbers to complete the square. Just list the square terms for each triplet:

$$(x - 2)^2 + (y + 8)^2 + (z - 3)^2 = 9 + 4 + 64 + 9 = 86$$

2 Vectors

Note: For the following discussion $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$

A. Length

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Note that a unit vector in the direction of any vector can be obtained by dividing the vector by its magnitude. For example, a unit vector in the direction of $\mathbf{a}$ is

$$\langle \frac{a_1}{|\mathbf{a}|}, \frac{a_2}{|\mathbf{a}|}, \frac{a_3}{|\mathbf{a}|} \rangle$$

Finally to find the vector $\overrightarrow{P_1P_2}$ from point $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ calculate

$$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

B. Standard Basis Vectors

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle$$

so

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$$

C. Force balance problems

To find the coordinates of a force, $\mathbf{F}$, always decompose it to its horizontal and vertical coordinates in the following way: if the acute angle between the vector and the horizontal is θ then

$$\text{horizontal component} = |\mathbf{F}| \cos\theta \begin{cases} \mathbf{i} & \text{if it points right} \\ -\mathbf{i} & \text{if it points left} \end{cases}$$

$$\text{vertical component} = |\mathbf{F}| \sin\theta \begin{cases} \mathbf{j} & \text{if it points up} \\ -\mathbf{j} & \text{if it points down} \end{cases}$$

To balance the forces

$$\text{Sum of all horizontal components} = 0$$
$$\text{Sum of all vertical components} = 0$$

D. Dot Product

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

where θ is the acute angle between the vectors. You can find the dot product in your calculator by typing

$$\texttt{dotP}([a_1, a_2, a_3], [b_1, b_2, b_3]).$$

1. $\mathbf{a}$ is orthogonal (perpendicular) to $\mathbf{b}$ if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.
2. Dot products are used to find the angle between two vectors as follows:

$$\theta = \cos^{-1} \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}.$$

3. Recall that the amount of work done by force $\mathbf{F}$ moving an object a distance $|\mathbf{d}|$ in the direction of vector $\mathbf{d}$ is

$$\text{Work} = \mathbf{F} \cdot \mathbf{d}.$$

4. Projections

$$\text{Scalar Projection} \qquad \text{comp}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}.$$

and

$$\text{Vector Projection} \qquad \text{proj}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} \mathbf{b}.$$

E. Cross Product

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\phi\, \mathbf{n}$$

where ϕ is the angle that $\mathbf{a}$ needs to be rotated to be in the same direction as $\mathbf{b}$, and $\mathbf{n}$ is a vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. The direction of $\mathbf{n}$ is obtained by the right hand rule. You can find the cross product in your calculator by typing

$$\texttt{crossP}([a_1, a_2, a_3], [b_1, b_2, b_3]).$$

1. $\mathbf{a} \times \mathbf{b}$ is orthogonal (perpendicular) to both $\mathbf{a}$ and $\mathbf{b}$.
2. $\mathbf{a}$ is parallel to $\mathbf{b}$ if and only if $\mathbf{a} \times \mathbf{b} = 0$.
3. Recall that the torque applied by force $\mathbf{F}$ on a wrench with length $|\mathbf{r}|$ is

$$\text{Torque} \quad \tau = \mathbf{r} \times \mathbf{F}.$$

4. The area of the parallelogram defined by $\mathbf{a}$ and $\mathbf{b}$ is

$$|\mathbf{a} \times \mathbf{b}|$$

3 Lines & Planes

Lines: Given a point on the line (x_0, y_0, z_0) and a parallel vector $\langle m_1, m_2, m_3 \rangle$ the various equations that describe a line in three dimensions are:

$$\begin{aligned}
&\text{Vector equation} && \langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle m_1, m_2, m_3 \rangle \\
&\text{parametric equations} && \begin{cases} x = x_0 + m_1 t \\ y = y_0 + m_2 t \\ z = z_0 + m_3 t \end{cases} \\
&\text{symmetric equations} && \frac{x - x_0}{m_1} = \frac{y - y_0}{m_2} = \frac{z - z_0}{m_3}
\end{aligned}$$

Here are all possible problems of finding the equation of a line with their solutions

1. *Given:* two points. *Solution:* Form the vector between the two points which will give the parallel vector; use any one of the two points.
2. *Given:* a point and a parallel line. *Solution:* use the parallel vector of the given line and the point.
3. *Given:* a point and a perpendicular plane. *Solution:* the perpendicular vector of the given plane is parallel to the line.

Planes: Given a point on the plane (x_0, y_0, z_0) and a perpendicular vector $\langle A, B, C \rangle$ the equation that describes a plane in three dimensions is:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Here are all possible problems of finding the equation of a plane with their solutions

1. *Given:* three points. *Solution:* Form two vectors between the three points and take the cross product to find the perpendicular vector; use any one of the three points.
2. *Given:* a point and a line on the plane. *Solution:* use the parallel vector of the given line; form a vector from the point on the line to the point on the plane; take the cross product to get a perpendicular vector.
3. *Given:* a point and a parallel plane. *Solution:* the perpendicular vector of the given plane is perpendicular to plane in question.

4 Parametric Curves

Curves for which x and y are given as functions of t:

$$x = f(t) \qquad y = g(t), \quad a \leq t \leq b$$

Examples of parametric curves: The limits on t are extremely important.

1. Circle or arc(part of circle) or radius r

$$\begin{aligned} x &= r\cos t \\ y &= r\sin t, \end{aligned}$$

   where $0 \leq t \leq 2\pi$ for a circle and $t_1 \leq t \leq t_2$ for an arc.

2. Line segment from point (x_1, y_1, z_1) to (x_2, y_2, z_2):

$$\begin{aligned} x &= (1-t)x_1 + tx_2 \\ y &= (1-t)y_1 + ty_2, \end{aligned}$$

   where $0 \leq t \leq 1$

3. The curve of the function $y = f(x)$ for $a \leq x \leq b$

$$\begin{aligned} x &= t \\ y &= f(t), \end{aligned}$$

   where $a \leq t \leq b$

4. The curve of the function $x = g(y)$ for $c \leq y \leq d$

$$\begin{aligned} x &= g(t) \\ y &= t, \end{aligned}$$

   where $c \leq t \leq d$

Calculus of parametric curves:

1. Derivatives
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
Note: When computing the slope of a tangent line at a point, always find the t that corresponds to that point and plug it into the slope.

2. Arc length, L, of the curve $x = f(t),\ y = g(t)$ where $a \leq t \leq b$
$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt.$$

Note: If the curve is given by $y = f(x),\ a \leq x \leq b$ the arc length is given by:
$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

Alternatively, if the curve is given by $x = g(y),\ c \leq y \leq d$ the arc length is given by:
$$L = \int_c^d \sqrt{(g'(y))^2 + 1}\, dy.$$