Review Notes: Exam 4

1 Modeling with Differential Equations

A. GENERAL AND SPECIFIC SOLUTIONS

1. If given a solution to verify for a differential equation, simply plug it into the DE and ensure the equality remains valid.

   For example, to show $y = \sin x + x^2$ is a solution to $y'' + y = x^2 + 2$, determine y'' and add it to y, verifying that the sum is $x^2 + 2$.

2. If given an initial condition, then you will be verifying a specific solution. This requires the additional step of plugging the initial condition into the proposed solution to make sure that it not only satisfies the DE, but also that it satisfies the solution at the specific points.

B. ELECTRICAL CIRCUITS

1. Electrical Circuits lend themselves to modeling with differential equations.
2. Kirchoff's Law: Below L is the inductance (measured in Henries), R is the resistance (measured in Ohms), C is the capacitance (measured in Coulombs), E is the battery/generator voltage (measured in Volts):

   a. For RL circuits, your unknown is the current $i(t)$. The equation is

   $$L\frac{di}{dt} + Ri = E$$

   with initial condition $i(0)$ =initial current.

   b. For RC circuits, your unknown is the charge $q(t)$. The equation is

   $$R\frac{dq}{dt} + \frac{q}{C} = E$$

   with initial condition $q(0)$ =initial charge.

   Note that you may have to use the relationship $i = \frac{dq}{dt}$.

   Also, for steady state charge current take the limit as $t \to \infty$. Note that the limit may NOT be a number but a function. In the latter case, the limit eliminates terms of the form e^{-kt}, where k is a positive constant.

2 Direction Fields

A. The Direction Field, or Slope Field, is a graphical representation of the "output" of a differential equation, thought of as a function assigning to each point in the xy-plane a slope. For the equation

$$\frac{dy}{dx} = f(x, y)$$

this slope is obtained by taking arbitrary values for x and y and plugging them into $f(x, y)$.

1. Starting at any point in the plane and following these instantaneous rates of change will construct a specific solution to the differential equation, just like a numerical initial condition.

3 Euler's Method

A. Euler's Method numerically approximates the value of a solution $y_n \approx y(x_n)$ to the differential equation

$$\frac{dy}{dx} = f(x, y)$$

given an initial condition $y(x_0) = y_0$, provided that we can "step" from y_0 to y_n in discrete steps of size h.

1. For example, to determine $y(1.3)$ given $y(1)$, you could use three steps of size $h = 0.1$.

B. Use the formulae

$$x_{n+1} = x_n + h \qquad y_{n+1} = y_n + hf(x_n, y_n)$$

Make sure that you use the "old" values in y_{n+1}.

4 Separable Equations

Equations of the form(s)

$$\frac{dy}{dx} = \text{(term involing only } x\text{)(term involing only } y\text{)}$$
$$\frac{dy}{dx} = \text{(term involing only } x\text{)}$$
$$\frac{dy}{dx} = \text{(term involing only } y\text{)}$$

A. To solve isolate all x's on one side of the equation and all y's on the other side.

B. Integrate both sides and place arbitrary constants on one side of the equation for simplicity.

5 Exponential Growth and Decay

A. $\dfrac{dy}{dx} = ky,\ y(0) = y_0.$

1. Solution is $y(t) = y_0 e^{kt}$.
2. $k > 0$ means exponential growth.
3. $k < 0$ means exponential decay.

B. Radioactive Decay

1. Mathematically a simple exponential decay.
2. Physicists prefer half-life, t_h to decay constants; half-life helps you find the constant k from the following formula:

$$k = \frac{\ln(0.5)}{t_h}$$

6 Newton's Law of Cooling/Heating

A. The equation that gives the temperature T (at any time t) of an object with initial temperature T_0 placed in a room with temperature T_{amb} is:

$$\frac{dT}{dt} = k(T - T_{amb}),$$

where k a constant to be determined later.

B. Using separation of variables the solution is

$$T = T_{amb} + (T_0 - T_{amb})e^{kt}$$

To find k a problem usually states the following:

1. The temperature after $t_1$minutes/hours is T_1. In that case plug the given quantities to the solution.
2. When the temperature is T_2 the object cools/heats at a rate of r degrees per minute/hour. In that case plug the given quantities into the differential equation. Do NOT forget to put a minus sign on the rate if the object cools.

7 Linear Equations

A. Equations of the form

$$y' + p(x)y = q(x)$$

Note that the equation may NOT be in this form. You may have to manipulate it to bring it to that form. As a first step, always divide by the coefficient of y'.

B. First, let

$$\mu = \int p(x)dx$$

Note that you do not need a constant of integration here. The solution of the equation is

$$y = e^{-\mu}\left(\int e^{\mu}q(x)dx + C\right)$$

8 Sequences

A. Lists of numbers $\{a_n\} = a_1, a_2, a_3, \ldots$

B. Sequence converges if the terms approach a finite value far into the set

$$\lim_{n\to\infty} a_n = L$$

C. Other properties of sequences:

1. Monotonic: always increasing or always decreasing.
2. Bounded:
   a. Bounded Above: There is a number greater than or equal to every term.
   b. Bounded Below: There is a number less than or equal to every term

   RESULT: If increasing and bounded above, then convergent. Same if decreasing and bounded below.

9 Series

A. INFINITE sum of the terms of a sequence.

B. Series convergence means that all of the terms added together give a finite constant

$$\sum_{n=0}^{\infty} a_n = S.$$

C. TEST FOR DIVERGENCE: If the terms of the sequence converge to a non-zero value, or diverge, then the series diverges. That is

If $\lim\limits_{n\to\infty} a_n = L \neq 0$ or $\lim_{n\to\infty} a_n$ DNE, then $\sum_{n=0}^{\infty} a_n$ DNE

D. GEOMETRIC SERIES

To test whether a series is geometric, perform the following:

- divide any two consecutive terms (the later term by the earlier)
- repeat for a different pair of consecutive terms
- if the result of the above steps is the same, the series is geometric

Call the ratio r. The geometric series converges if $|r| < 1$ to $\frac{a}{1-r}$. If $|r| \geq 1$ the series diverges. Formally, what we saw above is that every new term of the series is formed by multiplying the old term by r. So if we start with the term a the series can be written as

$$a + ar + ar^2 + ar^3 + \ldots = \sum_{n=0}^{\infty} ar^n$$

RECALL that the geometric series is one of a few which we can evaluate.

E. ABSOLUTE CONVERGENCE

1. If $\sum\limits_{n=1}^{\infty} |a_n|$, (i.e. the series composed of the absolute values of the terms of the original series) converges, then $\sum\limits_{n=1}^{\infty} a_n$ is said to be "Absolutely Convergent".
2. If a series is Absolutely Convergent, then it is Convergent.
3. Ratio Test

$$\text{If } \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \text{ then } \sum_{n=1}^{\infty} a_n \text{ converges absolutely}$$

$$\text{If } \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \text{ then } \sum_{n=1}^{\infty} a_n \text{ diverges}$$

Note that the Ratio Test is inconclusive if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

10 Power Series

A. Functions of x.

B. $f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ centered at 0 or

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$ centered at a

C. Determine values of x for which power series converges, called the interval of convergence which has a radius R about a, with the Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

1. If the series converges only if $x = a$, then $R = 0$.
2. If the series converges for all x, then $R = \infty$.
3. If the series converges for some values of x, then $R > 0$ with the series converging in the interval $(a - R, a + R)$. (Here the endpoints are NOT included.)

11 Functions Written as Series Sums

A. Use algebra to manipulate problems into a transformation of the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

Note that you can substitute anything for x.

B. Note that you can use derivatives or integrals to express more complicated functions as series. Here are a couple of examples:

$$\begin{aligned} -\ln(1-x) &= \int \frac{1}{1-x}\,dx = \int (1 + x + x^2 + \ldots)\,dx \\ &= x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \end{aligned}$$

or

$$\begin{aligned} \arctan(x) &= \int \frac{1}{1+x^2}\,dx = \int \frac{1}{1-(-x^2)}\,dx = \int (1 + (-x^2) + (-x^2)^2 + \ldots)\,dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots \end{aligned}$$

12 Taylor and MacLaurin Series

A. Used to approximate values of functions which can be written as power series.

B. Taylor Series of f about a

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

C. MacLaurin Series of f is a Taylor Series about $a = 0$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

D. Find the first four or five derivatives, evaluate them at a so that you can determine the coefficients. After you have constructed the respective terms, observe the pattern and write the n-th term.