SM365 – Practice for Exam 2 Solutions

1. (i) Since the first entry of the first row is zero, we swap with the first row with a non-zero entry. Thus:

$$\left[\begin{array}{cccc|c} 3 & 0 & 3 & -4 & 7 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & 1 & 3 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 3 & 0 & 3 & -4 & 7 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & \frac{10}{3} & \frac{11}{3} \\ 0 & 3 & -1 & \frac{17}{3} & \frac{4}{3} \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 3 & 0 & 3 & -4 & 7 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & \frac{7}{3} & \frac{11}{3} \\ 0 & 0 & -4 & \frac{8}{3} & \frac{4}{3} \end{array}\right]$$

One last pass yields:

$$\left[\begin{array}{cccc|c} 3 & 0 & 3 & -4 & 7 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & \frac{7}{3} & \frac{11}{3} \\ 0 & 0 & 0 & -\frac{20}{3} & -\frac{40}{3} \end{array}\right]$$

Back solving now yields:

$$-\frac{20}{3}x_4 = -\frac{40}{3} \Rightarrow x_4 = 2$$
$$-x_3 + \frac{7}{3}x_4 = \frac{11}{3} \Rightarrow x_3 = -\frac{11}{3} + \frac{14}{3} = 1$$
$$x_2 + x_3 + x_4 = 0 \Rightarrow x_2 = -3$$
$$3x_1 + 3x_3 - 4x_4 = 7 \Rightarrow x_1 = \frac{1}{3}(7 - 3 + 8) = 4$$

(ii) Initialize the row vector to $\mathbf{r} = [1\ 2\ 3\ 4]^T$. Among the values

$$|a_{r_1,1}| = 0,\ |a_{r_2,1}| = 3,\ |a_{r_3,1}| = 1,\ |a_{r_4,1}| = 2$$

the largest corresponds to r_2. We therefore interchange the first and second entries in the row vector to obtain

$$\mathbf{r} = [2\ 1\ 3\ 4]^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & -4 & 7 \\ 0 & 1 & 0 & \frac{10}{3} & \frac{11}{3} \\ 0 & 3 & -1 & \frac{17}{3} & \frac{4}{3} \end{array}\right]$$

Now,

$$|a_{r_2,2}| = 1,\ |a_{r_3,2}| = 1 \quad \text{and} \quad |a_{r_4,2}| = 3.$$

The larger value corresponds to r_4, therefore we interchange the second and fourth entries in the row vector to obtain

$$\mathbf{r} = [2\ 4\ 3\ 1]^T.$$

The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{cccc|c} 0 & 0 & \frac{4}{3} & -\frac{8}{9} & -\frac{4}{9} \\ 3 & 0 & 3 & -4 & 7 \\ 0 & 0 & \frac{1}{3} & \frac{13}{9} & \frac{29}{9} \\ 0 & 3 & -1 & \frac{17}{3} & \frac{4}{3} \end{array}\right]$$

Now,

$$|a_{r_3,3}| = \frac{4}{3} \quad \text{and} \quad |a_{r_4,3}| = \frac{1}{3}.$$

The larger value corresponds to r_3, , so there is no need to modify the contents of the row vector. The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{cccc|c} 0 & 0 & \frac{4}{3} & -\frac{8}{9} & -\frac{4}{9} \\ 3 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & \frac{5}{3} & \frac{10}{3} \\ 0 & 3 & -1 & \frac{17}{3} & \frac{4}{3} \end{array}\right]$$

Since we are working with infinite-digit arithmetic the solution is the same as in (i).

(iii) Initialize the row vector to $\mathbf{r} = [1\ 2\ 3\ 4]^T$. Since

$$\max_{1\le j\le 4} |a_{1j}| = 1, \ \max_{1\le j\le 4} |a_{2j}| = 4, \ \max_{1\le j\le 4} |a_{3j}| = 2 \text{ and } \max_{1\le j\le 4} |a_{4j}| = 3,$$

the scale vector is

$$\mathbf{s} = [1\ 4\ 2\ 3]^T.$$

Among the values

$$\frac{|a_{r_1 1}|}{s_{r_1}} = \frac{0}{1}, \ \frac{|a_{r_2 1}|}{s_{r_2}} = \frac{3}{4}, \ \frac{|a_{r_3 1}|}{s_{r_3}} = \frac{1}{2}, \ \frac{|a_{r_4 1}|}{s_{r_4}} = \frac{2}{3}$$

the largest corresponds to r_2. We therefore interchange the first and second entries in the row vector to obtain $\mathbf{r} = [2\ 1\ 3\ 4]^T$. The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & -4 & 7 \\ 0 & 1 & 0 & \frac{10}{3} & \frac{11}{3} \\ 0 & 3 & -1 & \frac{17}{3} & \frac{4}{3} \end{array}\right]$$

Now,

$$\frac{|a_{r_2 2}|}{s_{r_2}} = \frac{1}{1}, \ \frac{|a_{r_3 2}|}{s_{r_3}} = \frac{1}{2} \quad \text{and} \quad \frac{|a_{r_4 2}|}{s_{r_4}} = \frac{3}{3}.$$

The larger value corresponds to r_2, so there is no need to modify the contents of the row vector. The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & -4 & 7 \\ 0 & 0 & -1 & \frac{7}{3} & \frac{11}{3} \\ 0 & 0 & -4 & \frac{8}{3} & \frac{4}{3} \end{array}\right]$$

Now,

$$\frac{|a_{r_3 3}|}{s_{r_3}} = \frac{1}{2} \qquad \text{and} \qquad \frac{|a_{r_4 3}|}{s_{r_4}} = \frac{4}{3}.$$

The larger value corresponds to r_4. We therefore interchange the third and fourth entries in the row vector to obtain $\mathbf{r} = [2\ 1\ 4\ 3]^T$. The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & \frac{5}{3} & \frac{10}{3} \\ 0 & 0 & -4 & \frac{8}{3} & \frac{4}{3} \end{array}\right]$$

Since we are working with infinite-digit arithmetic the solution is the same as in (i).

2. This is problem 1 of section 3.3; please see the solution supplied on the web-site.

3. See the top of page 173 of the textbook.

4. See the middle to bottom of page 174 of the textbook.

5. See the bottom of page 175 and top of page 176 of the textbook.

6. We have

$$\|\mathbf{x}\|_2 = \sqrt{30} \quad \text{and} \quad \|\mathbf{x}\|_\infty = 4$$

7. We have

$$\|A\|_2 = 6.1732 \quad \text{and} \quad \|A\|_\infty = 6$$

8. $\kappa_\infty = 200$

9. (a) This is problem 14(a) of section 3.5. Please refer to the solutions on this web-site.

(b) With $\mathbf{b} = [-3 \quad -12 \quad 6]^T$, we find $P\mathbf{b} = [6 \quad -3 \quad -12]^T$. Now, forward substitution applied to $L\mathbf{z} = P\mathbf{b}_1$ yields

$$z_1 = 6,\ z_2 = -3 - \frac{1}{2}z_1 = -6,\ z_3 = 12 - \frac{3}{2}z_1 - \frac{31}{11}z_2 = -\frac{45}{11}.$$

Next, back substitution applied to $U\mathbf{x} = \mathbf{z}$ gives

$$x_3 = \frac{z_3}{-45/11} = 1;\ x_2 = \frac{-6 - 5x_3}{11/2} = -2;\ \text{and}$$
$$x_1 = \frac{6 - 3x_2}{4} = 3.$$

10. Use your codes to confirm your answers.

11. We start by noting that

$$\begin{aligned} \mathbf{x}^{(k+1)} - \mathbf{x}^{(k)} &= T\mathbf{x}^{(k)} + \mathbf{c} - (T\mathbf{x}^{(k-1)} + \mathbf{c}) \\ &= T(\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}) \\ &= T^2(\mathbf{x}^{(k-1)} - \mathbf{x}^{(k-2)}) \\ &= \cdots \\ &= T^k(\mathbf{x}^{(1)} - \mathbf{x}^{(0)}) \end{aligned}$$

Therefore,

$$\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| = \|T^k(\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| = \|T^k\|\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|.$$

Now, let $m > k$. Then

$$\begin{aligned}
\|\mathbf{x}^{(m)} - \mathbf{x}^{(k)}\| &= \|\mathbf{x}^{(m)} - \mathbf{x}^{(m-1)} + \mathbf{x}^{(m-1)} - \mathbf{x}^{(m-2)} + \cdots + \mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| \\
&\leq \|\mathbf{x}^{(m)} - \mathbf{x}^{(m-1)}\| + \|\mathbf{x}^{(m-1)} - \mathbf{x}^{(m-2)}\| + \cdots + \|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| \\
&\leq (\|T\|^{m-1} + \|T\|^{m-2} + \cdots + \|T\|^k)\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| \\
&= \|T\|^k\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|(1 + \|T\| + \|T\|^2 + \cdots + \|T\|^{m-k-1})
\end{aligned}$$

Taking the limit as $m \to 1$, it follows that

$$\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \|T\|^k\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| \sum_{i=0}^{\infty} \|T\|^i = \frac{\|T\|^k}{1 - \|T\|}\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|.$$