

SM365 – PRACTICE FOR EXAM 2

1. Solve the given system using
 - (a) Gaussian elimination with no pivoting;
 - (b) Gaussian elimination with partial pivoting; and
 - (c) Gaussian elimination with scaled partial pivoting.

Compare the results obtained from each technique with the exact solution of the system.

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & -4 & 7 \\ 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & 1 & 3 & 6 \end{array} \right]$$

2. Show that $\|\cdot\|_\infty$ is a vector norm.
3. Prove the Cauchy-Buniakowski-Schwartz inequality, that is: Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then:

$$\left| \sum_{i=1}^n x_i y_i \right| \leq \|\mathbf{x}\|_2 \|\mathbf{y}\|_2.$$

4. Show that the l_2 - and l_∞ -vector norms are equivalent.
5. Let $\|\cdot\|_v$ be a vector norm. The natural norm associated with $\|\cdot\|_v$ is a matrix norm.
6. Compute the l_2 -norm and the l_∞ -norm for the vector

$$x = [2 \quad 1 \quad -3 \quad 4]^T$$

7. Compute the l_2 -norm and the l_∞ -norm for the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix}$$

8. Prove the following Theorem:

Let A be a non-singular matrix, $\tilde{\mathbf{x}}$ be an approximate solution to the linear system $A\mathbf{x} = \mathbf{b}$, $\mathbf{r} = A\tilde{\mathbf{x}} - \mathbf{b}$ and $\mathbf{e} = \tilde{\mathbf{x}} - \mathbf{x}$. Then, for any natural matrix norm $\|\cdot\|$,

$$\frac{1}{\|A\|} \|\mathbf{r}\| \leq \|\mathbf{e}\| \leq \|A^{-1}\| \|\mathbf{r}\|$$

and

$$\frac{1}{\|A\| \|A^{-1}\|} \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \leq \frac{\|\mathbf{e}\|}{\|\mathbf{x}\|} \leq \|A\| \|A^{-1}\| \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}$$

9. Compute κ_∞ for the matrix

$$A = \begin{bmatrix} 2.01 & 1.99 \\ 1.99 & 2.01 \end{bmatrix}$$

10. You are given the matrix A and vector \mathbf{b} :

$$A = \begin{bmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -3 \\ -12 \\ 6 \end{bmatrix}$$

(a) Using scaled partial pivoting during the factor step, find matrices L , U and P such that $LU = PA$.

(b) Solve the system $A\mathbf{x} = \mathbf{b}$.

11. Consider the linear system.

$$\begin{aligned} 5x + y - z &= 14 \\ x - 5y + 2z &= -9 \\ x - 2y + 10z &= -30 \end{aligned}$$

(a) Use four iterations for the Jacobi method to approximate the solution of this system, starting your initial guess at $x = 0$, $y = 0$, $z = 0$.

(b) Repeat part (a) for the Gauss-Seidel method.

12. Consider the iteration scheme $\mathbf{x}^{(k+1)} = T\mathbf{x}^{(k)} + \mathbf{c}$, and suppose that $\|T\| < 1$ for some natural matrix norm. Show that for any $\mathbf{x}^{(0)} \in \mathbb{R}^n$:

$$\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \frac{\|T\|^k}{1 - \|T\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$$

(*Hint:* Note that $\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| \leq \|T\|^k \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$; then consider the difference $\|\mathbf{x}^{(m)} - \mathbf{x}^{(k)}\|$, where $m > k$).