Solutions to Homework #2

1

(a) For $n > 1$,

$$\left|\frac{n-1}{n^3+2} - 0\right| = \frac{n-1}{n^3+2} < \frac{n}{n^3} = \frac{1}{n^2}.$$

Thus, $\frac{n-1}{n^3+2}$ converges to 0 with rate of convergence $O(1/n^2)$.

(b) Note that

$$\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n}) = \lim_{n\to\infty}\frac{(n+1)-n}{\sqrt{n+1}-\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$$

Because

$$|(\sqrt{n+1}-\sqrt{n}) - 0| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}},$$

it follows that $\sqrt{n+1}-\sqrt{n}$ converges to 0 with rate of convergence $O(1/\sqrt{n})$.

2

(a) From Taylor's Theorem, $e^x = 1 + x + \frac{1}{2}x^2e^\xi$ for some ξ between 0 and x. Hence,

$$\frac{e^x-1}{x} = 1 + \frac{1}{2}xe^\xi.$$

Because

$$\left|\frac{e^x-1}{x} - 1\right| = \frac{1}{2}|x|e^\xi < |x|$$

for all x satisfying $|x| < \ln 2$, it follows that

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1 \text{ with rate of convergence } O(x).$$

(b) From Taylor's Theorem, $\sin x = x - \frac{x^3}{6}\cos\xi$ for some ξ between 0 and x. Then,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6}\cos\xi$$

and

$$\left|\frac{\sin x}{x} - 1\right| = \frac{1}{6}|x^2\cos\xi| \le \frac{1}{6}x^2.$$

Finally,

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \text{ with rate of convergence } O(x^2).$$

3

The values in the following table suggest that $\frac{\sin x^2}{x^2}$ converges toward 1 more rapidly than $\frac{(\sin x)^2}{x^2}$.

x	$\dfrac{\sin x^2}{x^2}$	$\dfrac{(\sin x)^2}{x^2}$
1.000	0.84147098480790	0.70807341827357
0.100	0.99998333341667	0.99667110793792
0.010	0.99999999833333	0.99996666711111
0.001	0.99999999999983	0.99999966666671

To confirm this conclusion, note that by Taylor's Theorem,

$$\sin u = u - \frac{1}{6}u^3 \cos \xi,$$

for some ξ between 0 and u. Using the substitution $u = x^2$, we find

$$\sin x^2 = x^2 - \frac{1}{6}x^6 \cos \xi$$

for some ξ between 0 and x^2. Consequently,

$$\left| \frac{\sin x^2}{x^2} - 1 \right| = \frac{1}{6}x^4 |\cos \xi| \le \frac{1}{6}x^4.$$

Starting from $f(x) = (\sin x)^2$, we find

$$f'(x) = 2\sin x \cos x = \sin 2x, \;\; f''(x) = 2\cos 2x, \;\; f'''(x) = -4\sin 2x,$$

and $f^{(4)}(x) = -8\cos 2x$. Therefore,

$$(\sin x)^2 = x^2 - \frac{1}{3}x^4 \cos 2\xi$$

for some ξ between 0 and x, and

$$\left| \frac{(\sin x)^2}{x^2} - 1 \right| = \frac{1}{3}x^2 |\cos 2\xi| \le \frac{1}{3}x^2.$$

Finally,

$$\frac{\sin x^2}{x^2} = 1 + O(x^4) \qquad \text{and} \qquad \frac{(\sin x)^2}{x^2} = 1 + O(x^2).$$

4

(a) Suppose $\alpha_n = \alpha + O(1/n^b)$. Then, there exists a constant λ such that for sufficiently large n, $|\alpha_n - \alpha| \le \lambda \frac{1}{n^b}$. Because $a < b$, it follows that $n^a < n^b$ and $\frac{1}{n^a} > \frac{1}{n^b}$ for all $n > 1$. Thus,

$$|\alpha_n - \alpha| \le \lambda \frac{1}{n^b} < \lambda \frac{1}{n^a},$$

and $\alpha_n = \alpha + O(1/n^a)$.

(b) Suppose $f(x) = L + O(x^b)$. Then, there exists a constant K such that for all sufficiently small x, $|f(x) - L| \le K|x|^b$. Because $a < b$, it follows that for all $|x| \le 1$, $|x|^b \le |x|^a$. Thus, for sufficiently small x,

$$|f(x) - L| \le K|x|^b \le K|x|^a,$$

and $f(x) = L + O(x^a)$.

7

(a) If

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = 0,$$

then the numerator approaches zero faster than the denominator. In order to achieve a nonzero limit, we must increase the power in the denominator. Therefore, the order of convergence must be greater than α.

(b) If

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} \to \infty,$$

then the denominator approaches zero faster than the numerator. In order to achieve a nonzero limit, we must decrease the power in the denominator. Therefore, the order of convergence must be less than α.

10

Because the values in the third column of the following table are increasing with n, the evidence suggests that the sequence does not have order of convergence $\alpha = 1.618$, but rather that the order of convergence is less than 1.618. Because the values in the fourth column appear to be approaching a constant, these values suggest that the sequence is converging to $4/3$ with order of convergence $\alpha = 1$.

n	p_n	$\lvert p_n - 4/3\rvert / \lvert p_{n-1} - 4/3\rvert^{1.618}$	$\lvert p_n - 4/3\rvert / \lvert p_{n-1} - 4/3\rvert$
1	1.49866409858002		
2	1.49735399779221	3.01718763541581	0.99207588021590
3	1.42880197733534	1.77891367138598	0.58205253781266
4	1.40109291538955	3.03079120639280	0.70975745769255
5	1.37649367605146	3.36181849329742	0.63696294174768
6	1.36134574557313	4.52671513900300	0.64903127444432
7	1.35103448250088	5.75689539760301	0.63190377951100
8	1.34447985069507	7.61855893491390	0.62970586012393

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Note

$$
\begin{aligned}
x_{n+1} - \frac{1}{a} = x_n(2 - ax_n) - \frac{1}{a} &= -ax_n^2 + 2x_n - \frac{1}{a} \\
&= -a\left(x_n^2 - \frac{2}{a}x_n + \frac{1}{a^2}\right) = -a\left(x_n - \frac{1}{a}\right)^2.
\end{aligned}
$$

Thus,

$$
\lim_{n\to\infty} \frac{|x_{n+1} - \frac{1}{a}|}{|x_n - \frac{1}{a}|^2} = \lim_{n\to\infty} a = a.
$$

Consequently, $x_n \to \frac{1}{a}$ with order of convergence $\alpha = 2$ and asymptotic error constant $\lambda = a$.

18

Let $f(x) = e^x$. Then $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for all n. Therefore, by Taylor's Theorem with $x_0 = 0$,

$$
\begin{aligned}
e^x &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1} \\
&= 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^{\xi},
\end{aligned}
$$

for some ξ between 0 and x.