Solutions to Homework #5

1

(c) Let $f(x) = e^{-x} - x$. Because f is continuous on $[0, 1]$ with $f(0) = 1 > 0$ and $f(1) = e^{-1} - 1 \approx -0.632 < 0$, the Intermediate Value Theorem guarantees that there exists a $p \in (0, 1)$ such that $f(p) = 0$. To start the bisection method, take $(a_1, b_1) = (0, 1)$. The midpoint of this first interval, and our first approximation to the location of the root, is

$$p_1 = \frac{a_1 + b_1}{2} = \frac{0 + 1}{2} = 0.5.$$

Note that $f(p_1) \approx 0.107 > 0$. Since $f(a_1)$ and $f(p_1)$ are of the same sign, the Intermediate Value Theorem tells us that the root lies between p_1 and b_1. With $(a_2, b_2) = (a_1, p_1) = (0.5, 1)$, our second approximation to the location of the root is

$$p_2 = \frac{a_2 + b_2}{2} = \frac{0.5 + 1}{2} = 0.75.$$

Now $f(p_2) \approx -0.278 < 0$, which is of the opposite from $f(a_2)$. Hence, the Intermediate Value Theorem guarantees that the root is between a_2 and p_2, so we take $(a_3, b_3) = (a_2, p_2) = (0.5, 0.75)$. For the third iteration, we calculate

$$p_3 = \frac{a_3 + b_3}{2} = \frac{0.5 + 0.75}{2} = 0.625$$

and $f(p_3) \approx -0.0897 < 0$. Here, $f(a_3)$ and $f(p_3)$ are of opposite sign, which implies that the root lies between a_3 and p_3. Finally, we set $(a_4, b_4) = (0.5, 0.625)$.

4

Let $f(x) = 1 - \ln x$. Because f is continuous on $(2, 3)$ with $f(2) = 1 - \ln 2 \approx 0.307 > 0$ and $f(3) = 1 - \ln 3 \approx -0.0986 < 0$, the Intermediate Value Theorem guarantees that there exists a $p \in (2, 3)$ such that $f(p) = 0$. The following table summarizes the first five iterations of the bisection method starting from the interval $(a_1, b_1) = (2, 3)$.

n	p_n	$\lvert p_n - p\rvert$	$(b-a)/2^n$
1	2.50000	0.21828	0.50000
2	2.75000	0.03172	0.25000
3	2.62500	0.09328	0.12500
4	2.68750	0.03078	0.06250
5	2.71875	0.00047	0.03125

6

Let p denote the location of the true root, and let p_n denote the approximate location of the root produced by the nth iteration of the bisection method. From the proof of the bisection method convergence theorem, we know that

$$|p_n - p| \le \frac{b-a}{2^n}.$$

The bisection method sequence will therefore converge to within an absolute tolerance of ϵ provided

$$\frac{b-a}{2^n} < \epsilon.$$

Solving this last expression for n gives

$$n > \log_2 \frac{b-a}{\epsilon}.$$

11

Let $f(x) = x^3 - 13$. Since $f(2) = -5 < 0$ and $f(3) = 14 > 0$, we know there is a root on the interval $(a_1, b_1) = (2, 3)$. Using a convergence tolerance of $\epsilon = 5 \times 10^{-4}$, the bisection method yields

n	Enclosing Interval	Approximation
1	(2.000000,3.000000)	2.5000000000
2	(2.000000,2.500000)	2.2500000000
3	(2.250000,2.500000)	2.3750000000
4	(2.250000,2.375000)	2.3125000000
5	(2.312500,2.375000)	2.3437500000
6	(2.343750,2.375000)	2.3593750000
7	(2.343750,2.359375)	2.3515625000
8	(2.343750,2.351562)	2.3476562500
9	(2.347656,2.351562)	2.3496093750
10	(2.349609,2.351562)	2.3505859375
11	(2.350586,2.351562)	2.3510742188

Thus, $\sqrt[3]{13} \approx 2.35107$, with an error of at most 5×10^{-5}.

Code

```
function bisect(f,a,b,tol,n)
% Bisection method for solving the nonlinear
%equation  f(x)=0.
a0=a;
b0=b;
iter=0;
u=feval(f,a);
```

```
v=feval(f,b);
c=(a+b)*0.5;
err=abs(b-a)*0.5;
disp('___________________________________________________________
__________')
disp(' iter    a            b            c            f(c)       |b-
a|/2  ')
disp('___________________________________________________________
__________')
fprintf('\n')
if (u*v<=0)
   while (err>tol)&(iter<=n)
      w=feval(f,c);
      fprintf('%2.0f  %10.4f  %10.4f  %12.6f  %10.6f
%10.6f\n',iter,a,b,c,w,err)
      if (w*u<0)
         b=c;v=w;
      end
      if (w*u>0)
         a=c;u=w;
      end
      iter=iter+1;
      c=(a+b)*0.5;
      err=abs(b-a)*0.5;
   end
   if (iter>n)
      disp('  Method failed to converge')
   end
else
   disp('   The method cannot be applied f(a)f(b)>0')
end
```