Solutions to homework #8 (Section 3.3)

1.

In what follows, let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary n-vectors, and let α be an arbitrary real number.

(i): $\|\mathbf{x}\|_\infty \geq 0$

Since $|x_i| \geq 0$ for any real number x_i, it follows that

$$\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i| \geq 0.$$

(ii): $\|\mathbf{x}\|_\infty = 0$ if and only if $\mathbf{x} = \mathbf{0}$

If $\mathbf{x} = \mathbf{0}$, then $x_i = 0$ for each i. Therefore, $\max_{1 \leq i \leq n} |x_i| = 0$ and $\|\mathbf{x}\|_\infty = 0$. Conversely, if $\|\mathbf{x}\|_\infty = 0$, then $\max_{1 \leq i \leq n} |x_i| = 0$. This can happen only if $x_i = 0$ for each i, so $\mathbf{x} = \mathbf{0}$.

(iii): $\|\alpha\mathbf{x}\|_\infty = |\alpha| \, \|\mathbf{x}\|_\infty$

$$\|\alpha\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |\alpha x_i| = |\alpha| \max_{1 \leq i \leq n} |x_i| = |\alpha| \, \|\mathbf{x}\|_\infty.$$

(iv): $\|\mathbf{x}+\mathbf{y}\|_\infty \leq \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty$

$$\begin{aligned} \|\mathbf{x}+\mathbf{y}\|_\infty &= \max_{1 \leq i \leq n} |x_i + y_i| \leq \max_{1 \leq i \leq n} (|x_i| + |y_i|) \\ &\leq \max_{1 \leq i \leq n} |x_i| + \max_{1 \leq i \leq n} |y_i| = \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty. \end{aligned}$$

2.

(b) Let $\mathbf{x} = [\begin{array}{cccc} 2 & 1 & -3 & 4 \end{array}]^T$. Then

$$\|\mathbf{x}\|_2 = \sqrt{2^2 + 1^2 + (-3)^2 + 4^2} = \sqrt{30},$$

and

$$\|\mathbf{x}\|_\infty = \max\{|2|, |1|, |-3|, |4|\} = 4.$$

3.

(a)

To establish that $\|\cdot\|_1$ is a vector norm, we must show that $\|\cdot\|_1$ satisfies each of the four properties of the definition. In what follows, let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary n-vectors, and let α be an arbitrary real number.
(i): $\|\mathbf{x}\|_1 \geq 0$

Since $|x_i| \geq 0$ for any real number x_i, it follows that

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i| \geq 0.$$

(ii): $\|\mathbf{x}\|_1 = 0$ if and only if $\mathbf{x} = \mathbf{0}$

If $\mathbf{x} = \mathbf{0}$, then $x_i = 0$ for each i. Therefore, $\sum_{i=1}^{n} |x_i| = 0$ and $\|\mathbf{x}\|_1 = 0$. Conversely, if $\|\mathbf{x}\|_1 = 0$, then $\sum_{i=1}^{n} |x_i| = 0$. This can happen only if $x_i = 0$ for each i, so $\mathbf{x} = \mathbf{0}$.

(iii): $\|\alpha\mathbf{x}\|_1 = |\alpha|\,\|\mathbf{x}\|_1$

$$\|\alpha\mathbf{x}\|_1 = \sum_{i=1}^{n} |\alpha x_i| = |\alpha| \sum_{i=1}^{n} |x_i| = |\alpha|\,\|\mathbf{x}\|_1.$$

(iv): $\|\mathbf{x}+\mathbf{y}\|_1 \leq \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1$

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|_1 &= \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} (|x_i| + |y_i|) \\ &\leq \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1.\end{aligned}$$

(c)

Let x_k be such that $\|\mathbf{x}\|_\infty = |x_k|$. Then it immediately follows that

$$\|\mathbf{x}\|_\infty = |x_k| \leq \sum_{i=1}^{n} |x_i| = \|\mathbf{x}\|_1.$$

Similarly,

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i| \leq \sum_{i=1}^{n} |x_k| = n\|\mathbf{x}\|_\infty.$$

4.

Let $\mathbf{x}$ be any non-zero n-vector. Using the consistency property of the natural norm twice, we find

$$\|AB\mathbf{x}\|_v = \|A(B\mathbf{x})\|_v \leq \|A\| \, \|B\mathbf{x}\|_v \leq \|A\| \, \|B\| \, \|\mathbf{x}\|_v.$$

Therefore,

$$\frac{\|AB\mathbf{x}\|_v}{\|\mathbf{x}\|_v} \leq \|A\| \, \|B\| \quad \Rightarrow \quad \|AB\| \leq \|A\| \, \|B\|.$$

5

(b) Let $A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$. The characteristic polynomial associated with this matrix is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) = \det\left(\begin{bmatrix} 0.7-\lambda & 0.2 \\ 0.3 & 0.8-\lambda \end{bmatrix}\right) \\ &= (0.7-\lambda)(0.8-\lambda) - 0.06 = \lambda^2 - 1.5\lambda + 0.5. \end{aligned}$$

The eigenvalues of A are the roots of this polynomial: $\lambda = 0.5$ and $\lambda = 1$. Thus, $\sigma(A) = \{0.5, 1\}$.

(d) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix}$. The characteristic polynomial associated with this matrix is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 5 & -1-\lambda \end{bmatrix}\right) \\ &= (1-\lambda)\left[(3-\lambda)(-1-\lambda) - 5\right] \\ &= (1-\lambda)(\lambda-4)(\lambda+2). \end{aligned}$$

The eigenvalues of A are the roots of this polynomial: $\lambda = -2$, $\lambda = 1$ and $\lambda = 4$. Thus, $\sigma(A) = \{-2, 1, 4\}$.

6

(a) Let

$$A = \begin{bmatrix} 5 & -4 \\ -1 & 7 \end{bmatrix}.$$

Then

$$\|A\|_\infty = \max\{|5| + |-4|, |-1| + |7|\} = \max\{9, 8\} = 9.$$

To determine the l_2-norm, we first compute

$$A^T A = \begin{bmatrix} 5 & -1 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} 5 & -4 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 26 & -27 \\ -27 & 65 \end{bmatrix}.$$

The eigenvalues of this matrix are $\frac{1}{2}(91 \pm 3\sqrt{493})$. Hence,

$$\rho(A^T A) = \frac{1}{2}(91 + 3\sqrt{493}) \quad \text{and} \quad \|A\|_2 = \sqrt{\frac{1}{2}(91 + 3\sqrt{493})} \approx 8.87724.$$

(c) Let

$$A = \begin{bmatrix} 4 & -1 & -2 \\ 1 & 2 & -3 \\ 0 & 0 & 4 \end{bmatrix}.$$

Then

$$\|A\|_\infty = \max\{|4|+|-1|+|-2|, |1|+|2|+|-3|, |0|+|0|+|4|\} = \max\{7, 6, 4\} = 7.$$

To determine the l_2-norm, we first compute

$$A^T A = \begin{bmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ -2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -1 & -2 \\ 1 & 2 & -3 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 17 & -2 & -11 \\ -2 & 5 & -4 \\ -11 & -4 & 29 \end{bmatrix}.$$

The eigenvalues of this matrix are 35.72390, 2.94108 and 12.33502. Hence,

$$\rho(A^T A) = 35.72390 \quad \text{and} \quad \|A\|_2 = \sqrt{35.72390} \approx 5.97695.$$