Second-Derivative Test for Functions of Two Variables

1 Quadratic Approximation

Suppose we want to approximate the function $f(x, y)$ to second order around a point (x_0, y_0). That is, we want a polynomial approximation of the form

$$f(x,y) \approx A + B(x-x_0) + C(y-y_0) + \\ D(x-x_0)^2 + E(x-x_0)(y-y_0) + F(y-y_0)^2 \quad (1)$$

for constants A, B, C, D, E and F. For this to agree with our linear (tangent plane) approximation

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

we must have $A = f(x_0, y_0)$, $B = f_x(x_0, y_0)$ and $C = f_y(x_0, y_0)$.

Now differentiate (1) with respect to x to get

$$f_x(x,y) \approx B + 2D(x-x_0) + E(y-y_0).$$

Since the linear approximation applied to f_x is

$$f_x(x,y) \approx f_x(x_0,y_0) + f_{xx}(x_0,y_0)(x-x_0) + f_{xy}(x_0,y_0)(y-y_0),$$

we must take $D = \frac{1}{2}f_{xx}(x_0, y_0)$ and $E = f_{xy}(x_0, y_0)$. Finally, by considering the linear approximation to f_y we get $F = \frac{1}{2}f_{yy}(x_0, y_0)$. Thus, the quadratic approximation (1) is

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + \\ \frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2$$

2 Behavior of f at a Critical Point

Suppose now that (x_0, y_0) is a critical point of f, so that

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$$

Then our quadratic approximation becomes

$$f(x, y) - f(x_0, y_0) \approx$$
$$\frac{1}{2}[f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2],$$

so we need to concentrate on the behavior of

$$f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2 =$$
$$(x - x_0)^2 \left[f_{xx}(x_0, y_0) + 2f_{xy}(x_0, y_0)\frac{x - x_0}{y - y_0} + f_{yy}(x_0, y_0)\left(\frac{x - x_0}{y - y_0}\right)^2 \right] \quad (2)$$

for (x, y) near (x_0, y_0). Think of the expression in braces in (2) as a quadratic function of $t = (x - x_0)/(y - y_0)$:

$$q(t) = f_{xx}(x_0, y_0) + 2f_{xy}(x_0, y_0)t + f_{yy}(x_0, y_0)t^2.$$

Now $q(t)$ will never change sign as t varies if the roots of $q(t) = 0$ are complex. From the quadratic formula, this happens if

$$4f_{xy}(x_0, y_0)^2 - 4f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) < 0,$$

which is to say

$$f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 > 0.$$

In this case, $q(t)$ (and thus (2)) will be always positive if its constant term $f_{xx}(x_0, y_0) > 0$, and always negative if $f_{xx}(x_0, y_0) < 0$. So we have the conclusion:

If $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 > 0$, then f has a local minimum at (x_0, y_0) if $f_{xx}(x_0, y_0) > 0$, and a local maximum at (x_0, y_0) if $f_{xx}(p) < 0$.

Now suppose

$$f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 < 0.$$

Then the $q(t) = 0$ has two real roots, and so $q(t)$ changes sign. Thus, the expression (2) can be either positive or negative for (x, y) near (x_0, y_0), which means (x_0, y_0) is a saddle point of f:

If $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 < 0$, then f has a saddle point at p.