Labeled Posets, Iterated Integrals, and Nested Sums

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Outline

1. Yamamoto’s formalism
2. An algebra of 2-labeled posets
3. Integer-labeled posets and nested sums
4. From iterated integrals to nested sums
5. Extension to \((r+1)\)-labeled posets
Recently S. Yamamoto developed an elegant way of representing iterated integrals by 2-labeled posets. The iterated integrals in question are all over some subset of $[0, 1]^n$, and involve only the two forms

$$\omega_0(t) = \frac{dt}{t}, \quad \omega_1(t) = \frac{dt}{1 - t}.$$

Let $(X, \delta)$ be a 2-labeled poset, i.e., a finite partially ordered set $X$ with a function $\delta : X \to \{0, 1\}$. Call $(X, \delta)$ admissible if $\delta(x) = 1$ for all minimal $x \in X$ and $\delta(x) = 0$ for all maximal $x \in X$. For an admissible 2-labeled poset $(X, \delta)$, define the associated integral by

$$I(X) = \int_{\Delta(X)} \prod_{x \in X} \omega_{\delta(x)}(t_x), \quad (1)$$

where $\Delta(X) = \{(t_x)_{x \in X} \in [0, 1]^X \mid t_x < t_y \text{ if } x < y \text{ in } X\}$. 

**Outline**

- Yamamoto’s formalism
- An algebra of 2-labeled posets
- Integer-labeled posets and nested sums
- From iterated integrals to nested sums
- Extension to $(r + 1)$-labeled posets
We can represent a 2-labeled poset \((X, \delta)\) graphically by its Hasse diagram, with an open dot \(\circ\) for those elements \(x \in X\) with \(\delta(x) = 0\), and a closed dot \(\bullet\) for those \(x \in X\) with \(\delta(x) = 1\). For example, the 2-labeled poset \(X = \{x_1, x_2, x_3, x_4\}\) with \(x_1 > x_2 > x_4, x_1 > x_3 > x_4, \delta(x_1) = \delta(x_2) = 0, \) and \(\delta(x_3) = \delta(x_4) = 1\) has graphical representation

![](image)

and associated integral

\[
I(X) = \int_{t_1 > t_2 > t_4} \int_{t_1 > t_3 > t_4} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \frac{dt_3}{1 - t_3} \frac{dt_4}{1 - t_4}.
\]
2-labeled Posets and MZVs

Admissibility of \((X, \delta)\) guarantees convergence of the integral \(I(X)\). If \(X\) is a chain, then \(I(X)\) is just the well-known iterated integral representation of a multiple zeta value (MZV). For example,

\[
I\left( \begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array} \right) = \int_{t_1 > t_2 > t_3} \frac{dt_1}{t_1} \frac{dt_2}{1 - t_2} \frac{dt_3}{1 - t_3} = 
\]

\[
\int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1 - t_2} \int_0^{t_2} \frac{dt_3}{1 - t_3} = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \frac{t_2^{i+j-1}}{i} dt_2
\]

\[
= \int_0^1 \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \frac{t_1^{i+j-1}}{i(i+j)} dt_1 = \sum_{i,j \geq 1} \frac{1}{i(i+j)^2} = \zeta(2, 1).
\]
Our notation for MZVs is

\[ \zeta(k_1, \ldots, k_r) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}, \]

with \( k_1, \ldots, k_r \) positive integers and \( k_1 > 1 \) required for convergence. The representation of MZVs by iterated integrals is also encoded by the polynomial algebra \( \mathbb{Q}\langle x, y \rangle \), where \( x \) and \( y \) are noncommuting variables encoding \( \frac{dt}{t} \) and \( \frac{dt}{1-t} \) respectively; convergent integrals correspond to monomials that start with \( x \) and end with \( y \). The MZV \( \zeta(k_1, \ldots, k_r) \) is represented by the monomial \( x^{k_1-1} y \cdots x^{k_r-1} y \). Shuffle product gives a commutative operation \( \boxplus \) on \( \mathcal{H} = \mathbb{Q}\langle x, y \rangle \), which corresponds to multiplication of the integrals; indeed we can think of \( \zeta \) as a homomorphism \( (\mathcal{H}^0, \boxplus) \to \mathbb{R} \).
Yamamoto proved the following theorem, which follows easily from general properties of iterated integrals.

**Theorem**

Let $X$ be an admissible 2-labeled poset.

1. If $Y$ is another admissible 2-labeled poset, $I(X \uplus Y) = I(X)I(Y)$.

2. If $a, b \in X$ are incomparable, let $X_{a < b}$ be $X$ with the additional relation $a < b$. Then $I(X) = I(X_{a < b}) + I(X_{b < a})$.

3. If $X^\uparrow$ is $X$ with reversed order and new labeling function $\delta^\uparrow(x) = 1 - \delta(x)$, then $I(X^\uparrow) = I(X)$. 


Using the Properties

It is evident that combining two disjoint chains via Part 2 is equivalent to shuffle product in $\mathbb{Q}(x, y)$. For example,

$$I \left( \begin{array}{c} \bullet \\ \bullet \\ \end{array} \right) = I \left( \begin{array}{c} \bullet \\ \bullet \\ \end{array} \right) + 3I \left( \begin{array}{c} \bullet \\ \bullet \\ \end{array} \right) + 6I \left( \begin{array}{c} \bullet \\ \bullet \\ \end{array} \right)$$

is exactly parallel to

$$xy \sqcup x^2y = xyx^2y + 3x^2yxy + 6x^3y^2,$$

both showing that

$$\zeta(2)\zeta(3) = \zeta(2, 3) + \zeta(3, 2) + \zeta(4, 1).$$
Using the Properties cont’d

In fact, use of Part 2 allows us to write $I(X)$ as a sum of multiple zeta values for any admissible 2-labeled poset $X$. For example,

$$I\left(\begin{array}{c}
\bullet \\
\bullet
\end{array}\right) = I\left(\begin{array}{c}
\bullet
\end{array}\right) + I\left(\begin{array}{c}
\bullet
\end{array}\right) = \zeta(3, 1) + \zeta(2, 2).$$

By the way, the “sum theorem” for MZVs implies that the latter sum is $\zeta(4)$, and more generally that

$$I\left(\begin{array}{c}
\bullet \\
\bullet
\end{array}\right) = \zeta(n),$$

where the diagram has $n - k$ open dots and $k$ closed ones, for $2 \leq k \leq n - 2$. 
An advantage of Yamamoto’s formalism is that it makes it easy to see how various zeta functions can be written in terms of MZVs. For example, the Mordell-Tornheim sums

\[ T(n_1, n_2, \ldots, n_k; p) = \sum_{m_1, \ldots, m_k \geq 1} \frac{1}{m_1^{n_1} \cdots m_k^{n_k} (m_1 + \cdots + m_k)^p} \]

can be expressed as integrals associated with 2-labeled posets, as shown by the example

\[ I = \int_0^1 \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_2}{1-t_2} \int_0^{t_4} \frac{dt_1}{1-t_1} = \]

\[ \int_0^1 \frac{dt_4}{t_4} \sum_{i,j \geq 1} \frac{t_4^{i+j}}{i^2 j} = \sum_{i,j \geq 1} \frac{1}{i^2 j(i+j)} = T(2, 1; 1). \]
If we compare the shuffle product with the poset representing Mordell-Tornheim sums, it is easy to see that

\[ T(n_1, n_2, \ldots, n_k; p) = \zeta(x^p(x^{n_1-1}y \sqcup x^{n_2-1}y \sqcup \cdots \sqcup x^{n_k-1}y)), \quad (2) \]

which gives a succinct statement of how Mordell-Tornheim sums can be expanded into MZVs. For example,

\[
T(2, 1; 3) = \zeta(x^3(xy \sqcup y)) = \zeta(x^3(yxy + 2xy^2)) = \zeta(x^3yxy + 2x^4y^2) = \zeta(4, 2) + 2\zeta(5, 1).
\]

Before seeing Yamamoto’s idea, I’d been expanding Mordell-Tornheim sums into MZVs for years without recognizing equation (2).
Multiple Zeta-Star Values

The multiple zeta-star values

$$\zeta^*(k_1, \ldots, k_r) = \sum_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}},$$

can also be represented by integrals associated with 2-labeled posets, as in the example

$$I\left(\begin{array}{cc}
\circ & \\
\bullet & \\
\end{array}\right) = \int_0^1 \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_3}{1 - t_3} \int_0^{t_3} \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_1}{1 - t_1} =$$

$$\int_0^1 \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_3}{1 - t_3} \sum_{i \geq 1} \frac{1 - t_3^i}{i^2} = \int_0^1 \sum_{i, j \geq 1} \left[ \frac{t_4^{i-1}}{i^2 j} - \frac{t_4^{i+j-1}}{i^2(i + j)} \right] \frac{dt_4}{t_4}$$

$$= \sum_{i=1}^{\infty} \frac{1}{i^2} \sum_{j=1}^{\infty} \left[ \frac{1}{j^2} - \frac{1}{(i + j)^2} \right] = \sum_{i=1}^{\infty} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j^2} = \zeta^*(2, 2).$$
Using the Properties

Applying Part 2 of Yamamoto’s theorem to the representation of $T(2, 1; 1)$ gives

$$I\left(\begin{array}{c}
\bullet \\
\bullet
\end{array}\right) = I\left(\begin{array}{c}
\bullet \\

\end{array}\right) + 2I\left(\begin{array}{c}
\bullet \\
\bullet
\end{array}\right),$$

which just recovers the identity $T(2, 1; 1) = \zeta(2, 2) + 2\zeta(3, 1)$ well-known from partial fractions. But applying it to the representation of $\zeta^*(2, 2)$ gives

$$I\left(\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}\right) = I\left(\begin{array}{c}
\bullet \\
\bullet
\end{array}\right) + 4I\left(\begin{array}{c}
\bullet \\
\bullet
\end{array}\right),$$

implying $\zeta^*(2, 2) = \zeta(2, 2) + 4\zeta(3, 1)$; comparing with the more familiar $\zeta^*(2, 2) = \zeta(2, 2) + \zeta(4)$ gives $\zeta(3, 1) = \frac{1}{4}\zeta(4)$. 
An Algebra of 2-labeled Posets

We define a graded $\mathbb{Q}$-algebra $\mathcal{A}$ as follows. For each 2-labeled poset $X$, $\mathcal{A}$ has a generator $[X]$ in degree $\text{card } X$. We agree that $[X] = [Y]$ if $X$ and $Y$ are isomorphic as 2-labeled posets, and define the product by $[X][Y] = [X \pitchfork Y]$. If $\mathcal{A}^0 \subset \mathcal{A}$ is the subalgebra generated by admissible 2-labeled posets, then Yamamoto’s theorem implies that $I : \mathcal{A}^0 \to \mathbb{R}$ is a homomorphism.

Call a 2-labeled poset $X$ lower admissible if $\delta(x) = 1$ for every minimal element of $X$. Then the subspace $\mathcal{A}^1$ of $\mathcal{A}$ generated by lower admissible 2-labeled posets is a subalgebra, and in fact $\mathcal{A}^0 \subset \mathcal{A}^1 \subset \mathcal{A}$.
A Homomorphism From $\mathcal{A}$ to $\mathcal{H}$

Now define a $\mathbb{Q}$-linear map $J : \mathcal{A} \rightarrow \mathcal{H}$ by sending any chain \( \{x_1 > x_2 > \cdots > x_n\} \) to the monomial $a(x_1)a(x_2) \cdots a(x_n)$, where $a(x_i) = x$ if $\delta(x_i) = 0$ and $a(x_i) = y$ if $\delta(x_i) = 1$. In general define $J(P)$ as $\sum_i J(c_i)$, where $\sum_i c_i$ is the formal sum of all chains obtained by adding more relations to $P$.

Let $\mathcal{H}^1$ be the subspace of $\mathcal{H}$ generated by 1 and monomials ending in $y$. Note that $(\mathcal{H}^1, \sqcup)$ is a commutative algebra. Then $J$ restricted to $\mathcal{A}^1$ is a homomorphism $J : \mathcal{A}^1 \rightarrow \mathcal{H}^1$. Similarly, if $\mathcal{H}^0$ is the subspace of $\mathcal{H}^1$ generated by 1 and all monomials that start in $x$ and end in $y$, then $(\mathcal{H}^0, \sqcup)$ is an algebra and $J : \mathcal{A}^0 \rightarrow \mathcal{H}^0$ is a homomorphism such that $I(X) = \zeta(J(X))$. 
An advantage of expanding the domain of $J$ to $A^1$ is that we now have interesting combinatorial examples, e.g.,

\[
J(\bullet) = y, \quad J(\bullet) = y^2, \quad J(\bullet\bullet) = 2y^3,
\]

\[
J(\bullet\bullet) = 5y^4, \quad J(\bullet\bullet\bullet) = 16y^5,
\]

\[
J(\bullet\bullet\bullet) = 61y^6,
\]

where the series $1, 1, 2, 5, 16, 61, \ldots$ is that of Euler or up/down numbers.
Integer-labeled Posets and Nested Sums

We can encode nested sums with a formalism quite similar to Yamamoto’s. Let $P$ be a finite poset with a labeling function $\delta : P \to \mathbb{Z}^+$. Call $P$ admissible if $\delta(x) > 1$ for every maximal element of $P$. For admissible $P$, let $SOP(P)$ be the set of strictly order-preserving functions $\sigma : P \to \mathbb{Z}^+$, and define the sum $S(P)$ by

$$S(P) = \sum_{\sigma \in SOP(P)} \prod_{x \in P} \frac{1}{\sigma(x)\delta(x)}.$$

If $P$ is an admissible integer-labeled poset that is a chain, then $S(P)$ is a multiple zeta value; in fact

$$S(\{x_1 > x_2 > \cdots > x_n\}) = \zeta(\delta(x_1), \delta(x_2), \ldots, \delta(x_n)).$$
Properties of Nested Sums

We have the following counterpart of Yamamoto’s theorem for nested sums.

**Theorem**

Let $P$ be an admissible integer-labeled poset.

1. If $Q$ is another admissible integer-labeled poset, then
   \[ S(P \amalg Q) = S(P)S(Q) \]

2. If $a, b \in P$ are incomparable, let $P_{a < b}$ be $P$ with the additional relation $a < b$, and let $P_{a = b}$ be $P$ with $a$ and $b$ merged into a new element having label $\delta(a) + \delta(b)$. Then
   \[ S(P) = S(P_{a < b}) + S(P_{b < a}) + S(P_{a = b}). \]
An Algebra of Integer-labeled Posets

Now define a graded \(\mathbb{Q}\)-algebra \(\mathcal{B}\) as follows. For each integer-labeled poset \(X\), \(\mathcal{B}\) has a generator \([X]\); we assign it degree \(\sum_{x \in X} \delta(x)\). We agree that \([X] = [Y]\) if \(X\) and \(Y\) are isomorphic as integer-labeled posets, and define the product by \([X][Y] = [X \sqcup Y]\). If \(\mathcal{B}^0 \subset \mathcal{B}\) is the subalgebra generated by admissible integer-labeled posets, then the preceding theorem implies that \(S : \mathcal{B}^0 \rightarrow \mathbb{R}\) is a homomorphism.
A Homomorphism From $\mathcal{B}$ to $\mathcal{H}^1$

Now define a $\mathbb{Q}$-linear map $K : \mathcal{B} \to \mathcal{H}^1$ by sending any chain $\{x_1 > x_2 > \cdots > x_n\}$ to the monomial $x^{\delta(x_1)-1}y \cdots x^{\delta(x_n)-1}y$.

In general define $K(P)$ as $\sum_i K(c_i)$, where $\sum_i c_i$ is the formal sum of all chains obtained by adding more relations to $P$.

Evidently $K(\mathcal{B}^0) \subset \mathcal{H}^0$, where as above $\mathcal{H}^0$ is the subspace of $\mathcal{H}^1$ generated by 1 and all monomials of the form $xwy$.

To make $K : \mathcal{B} \to \mathcal{H}^1$ a homomorphism, we must give $\mathcal{H}^1$ the “quasi-shuffle” product $*$ defined recursively as follows:

$$w * 1 = 1 * w = w$$

for all monomials $w$, and

$$(x^{p-1}yu)(x^{q-1}v) = x^{p-1}y(u * x^{q-1}yv) + x^{q-1}y(x^{p-1}yu * v) + x^{p+q-1}y(u * v)$$

for all monomials $u, v$. Then $S(w) = \zeta(K(w))$ for $w \in \mathcal{H}^0$. 

Mapping 2-labeled Posets to Integer-labeled Posets

If we introduce an equivalence relation \( \approx \) on elements of \( \mathcal{A} \) by declaring \([X] \approx [X_{a<b}] + [X_{b<a}]\) for \( a, b \in X \) incomparable, then \( \mathcal{A}^1/\approx \) is isomorphic to \( (\mathcal{H}^1, \sqcup) \). Similarly, if we define \( \sim \) on \( \mathcal{B} \) by \([P] \sim [P_{a<b}] + [P_{b<a}] + [P_{a=b}]\) for \( a, b \in P \) incomparable, then \( \mathcal{B}/\sim \) is isomorphic to \( (\mathcal{H}^1, \ast) \). If we send the chain in \([X] \in \mathcal{A}^1/\approx \) corresponding to \( x_1^{i_1-1}y \cdots x_k^{i_k-1}y \) to the chain \([Y] = \{y_1 > \cdots > y_k\}\) in \( \mathcal{B}/\sim \) with \( \delta(y_j) = i_j \), then \( I([X]) = S([Y]) \), provided \([X] \in \mathcal{A}^0\), since both sides are equal to \( \zeta(i_1, \ldots, i_k) \). That is, the identity function from \( (\mathcal{H}^0, \sqcup) \) to \( (\mathcal{H}^0, \ast) \) is certainly not a homomorphism, but \( \zeta(u \sqcup v) = \zeta(u) \zeta(v) = \zeta(u \ast v) \).
Now it is well-known that the homomorphisms 
\( \zeta : (\mathfrak{H}^0, \sqcup) \to \mathbb{R} \) and \( \zeta : (\mathfrak{H}^0, \ast) \to \mathbb{R} \) can be extended to homomorphisms \( \zeta^{\sqcup} : (\mathfrak{H}^1, \sqcup) \to \mathbb{R}[T] \) and \( \zeta^{\ast} : (\mathfrak{H}^1, \ast) \to \mathbb{R}[T] \) by introducing an indeterminate \( T \) to represent \( \zeta(y) \). The superscripts are necessary because these extensions clearly do not agree: since \( y \sqcup y = 2y^2 \) while \( y \ast y = 2y^2 + xy \), we have \( \zeta^{\sqcup}(y^2) = \frac{1}{2} T^2 \) but \( \zeta^{\ast}(y^2) = \frac{1}{2} T^2 - \frac{1}{2} \zeta(2) \). But it is natural to ask if the identity function on \( \mathfrak{H}^0 \) can be extended to a linear function \( \mu : \mathfrak{H}^1 \to \mathfrak{H}^1 \) such that \( \zeta^{\sqcup}(u) = \zeta^{\ast}(\mu(u)) \) for \( u \in \mathfrak{H}^1 \). It appears so: and in fact, it appears \( \mu \) can be defined recursively on the filtration

\[
\mathfrak{H}^0 \subset \mathfrak{H}^0 + y\mathfrak{H}^0 \subset \mathfrak{H}^0 + y\mathfrak{H}^0 + y^2\mathfrak{H}^0 \subset \cdots \subset \mathfrak{H}^1
\]

by \( \mu(y \sqcup u) = y \ast \mu(u) \).
We note that Yamamoto’s formalism can be extended to iterated integrals of the forms

\[ \omega_\alpha(t) = \frac{dt}{\alpha^{-1} - 1} = \frac{\alpha dt}{1 - \alpha t}, \]

where \( \alpha \) is an \( r \)th root of unity. Here one must replace 2-labeled posets by \((r + 1)\)-labeled posets, where the labels come from the set \( \{0, 1, \epsilon, \ldots, \epsilon^{r-1}\} \), where \( \epsilon \) is a primitive \( r \)th root of unity. An \((r + 1)\)-labeled poset is admissible if minimal elements of \( X \) don’t have label 0 and maximal elements of \( X \) don’t have label 1. Then equation (1) carries over to this case, and one has Yamamoto’s theorem (except that one must drop the part about generalized duality).
Nested Sums for $r$th Roots of Unity

The generalization of MZVs that appears in this case are multiple polylogarithms evaluated at roots of unity: if we define

$$\text{Li}_{i_1,\ldots,i_k}(z_1,\ldots,z_k) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{i_1} \cdots n_k^{i_k}},$$

then $I(X)$ corresponding to the chain whose $\delta$’s are given by the string

$$(\omega_0)^{i_1-1}\omega_{\alpha_1} \cdots (\omega_0)^{i_k-1}\omega_{\alpha_k},$$

for $\alpha_1,\ldots,\alpha_k \in \{1, \epsilon, \epsilon^2, \ldots, \epsilon^{r-1}\}$ with $\alpha_1 \neq 1$ has value

$$\text{Li}_{i_1,\ldots,i_k}(\alpha_1, \frac{\alpha_2}{\alpha_1}, \ldots, \frac{\alpha_k}{\alpha_{k-1}}).$$
The Case $r = 2$

The case $r = 1$ is just that treated above. The case $r = 2$ applies to alternating MZVs like

$$\zeta(\bar{2}, 1, \bar{3}) = \sum_{l > m > n \geq 1} \frac{(-1)^{l+n}}{l^2 mn^3}$$

and gives nice “diagrammatic” evaluations of them. Shuffle-product formulas for generalized Mordell-Tornheim sums of this type, e.g.,

$$\sum_{n,m \geq 1} \frac{(-1)^n}{n^2 m(n + m)^2},$$

(similar to equation (2) above) are easily obtained.