Motion in an Inverse-Square Central Force Field

1 Central Forces and Angular Momentum

Suppose we have a *central force,* that is, a force that depends only on the distance to the origin and is directed either toward or away from the origin. Then we can write the force as

$$\vec{F} = f(r)\vec{r} \tag{1}$$

where $f(r)$ is a scalar function (We write $r = \sqrt{x^2 + y^2 + z^2}$ for the length of the vector $\vec{r}$: of course this is just the distance to the origin). For a particle of mass m whose position as a function of time is given by $\vec{r}(t)$, we define its *angular momentum* by

$$\vec{L}(t) = m\vec{r}(t) \times \vec{r}'(t)$$

Fact: If a particle moves subject to a central force, its angular momentum is constant.
Proof: Using the product rule, we compute the derivative of $\vec{L}(t)$ as

$$\vec{L}'(t) = m\vec{r}'(t) \times \vec{r}'(t) + m\vec{r}(t) \times \vec{r}''(t) = m\vec{r}(t) \times \vec{r}''(t)$$

since $\vec{r}'(t)$ crossed with itself is the zero vector. But Newton's second law says that

$$\vec{F} = m\vec{r}''(t),$$

so our equation is

$$\vec{L}'(t) = \vec{r}(t) \times \vec{F}.$$

But then, using equation (1), we have

$$\vec{L}'(t) = \vec{r}(t) \times f(r)\vec{r}(t) = \mathbf{0}$$

and so $\vec{L}(t)$ must be a constant vector.

The constancy of the vector $\vec{L}$ has two important consequences.

Consequence 1: The motion takes place in a plane containing the origin. This is because the vector $\vec{r}$ must always be perpendicular to the constant vector $\vec{L}$.

Consequence 2: The radius vector $\vec{r}$ sweeps out areas at a constant rate. (This is Kepler's first law in the context of planetary motion.)

Let ΔA be the area swept out by $r(t)$ in the time interval Δt. Then

$$\Delta A = \frac{1}{2}|\vec{r}(t) \times \vec{r}(t+\Delta t)| \approx \frac{1}{2}|\vec{r}(t) \times (\vec{r}(t) + \vec{r}'(t)\Delta t| = \frac{1}{2}|\vec{r}(t) \times \vec{r}'(t)|\Delta t$$

so that

$$\frac{\Delta A}{\Delta t} \approx \frac{1}{2}|\vec{r}(t) \times \vec{r}'(t)| = \frac{L}{2m},$$

where $L = |\vec{L}|$ (which is constant). In the limit,

$$\frac{dA}{dt} = \frac{L}{2m}.$$

2 Inverse-Square Force

Suppose now that $f(r) = -K/r^3$, where K is a constant. (Note that the force is attractive if $K > 0$ and repulsive if $K < 0$.) In this case $\vec{F}$ is called an inverse-square force since

$$\vec{F} = -\frac{K}{r^3}\vec{r} = -\frac{K}{r^3}r\hat{\mathbf{r}} = -\frac{K}{r^2}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is the unit vector in the direction of $\vec{r}$. In the case of planetary motion, $K = GM$, where G is the universal gravitational constant and M is the mass of the sun. In the case of motion in the electric field of a static charged particle, $K = Cq$, where C is a universal constant and q is the particle charge: in this case K can be negative.

From the product rule

$$\vec{L} = m\vec{r} \times (r\hat{\mathbf{r}})' = mr\hat{\mathbf{r}} \times (r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}') = mr^2\hat{\mathbf{r}} \times \hat{\mathbf{r}}',$$

and since $\vec{F} = -Kr^{-2}\hat{\mathbf{r}}$, we have

$$\vec{F} \times \vec{L} = -Kr^{-2}\hat{\mathbf{r}} \times (mr^2\hat{\mathbf{r}} \times \hat{\mathbf{r}}') = -Km\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{r}}').$$

Now use the vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \tag{2}$$

to write this as

$$\vec{F} \times \vec{L} = -Km((\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')\hat{\mathbf{r}} - (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}').$$

But $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$, and it follows from differentiating this equation that $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = 0$. Hence

$$\vec{F} \times \vec{L} = Km\hat{\mathbf{r}}'. \tag{3}$$

Since

$$\frac{d}{dt}\left(m\vec{r}' \times \vec{L}\right) = m\vec{r}'' \times \vec{L} = \vec{F} \times \vec{L},$$

equation (3) implies that

$$m\vec{r}' \times \vec{L} = Km\hat{\mathbf{r}} + \vec{c},$$

for some vector constant $\vec{c}$. Dot both sides with $\vec{r}$ to get

$$\vec{r} \cdot (m\vec{r}' \times \vec{L}) = Kmr + \vec{r} \cdot \vec{c}. \tag{4}$$

On the other hand, the vector identity (2) implies that

$$m\vec{r}' \times \vec{L} = m^2\vec{r}' \times (\vec{r} \times \vec{r}') = m^2|\vec{r}'|^2\vec{r} - m^2(\vec{r}' \cdot \vec{r})\vec{r}'$$

and thus

$$\vec{r} \cdot (m\vec{r}' \times \vec{L}) = m^2r^2|\vec{r}'|^2 - m^2(\vec{r}' \cdot \vec{r})^2. \tag{5}$$

Now

$$|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a} \times \mathbf{b}|^2$$

for any vectors $\mathbf{a}$ and $\mathbf{b}$, so equation (5) gives

$$\vec{r} \cdot (m\vec{r}' \times \vec{L}) = m^2|\vec{r}' \times \vec{r}|^2 = L^2.$$

Compare this with equation (4) to get

$$L^2 = Kmr + \vec{r} \cdot \vec{c}. \tag{6}$$

Now choose polar coodinates in the plane of motion so that $\vec{c}$ points in the direction of the positive x-axis. Then equation (6) is

$$L^2 = Kmr + cr\cos\theta$$

(where $c = |\vec{c}|$), or

$$r = \frac{L^2}{c\cos\theta + Km}. \tag{7}$$

Equation (7) is always a conic section (ellipse, parabola or hyperbola). It is the equation of an ellipse if $c < Km$ (and a circle if $c = 0$). If $Km = c$ it is a parabola, and for $Km < c$ it is an hyperbola (in particular, we always get hyperbolic orbits for a repulsive force). If the path of a particle is a closed orbit, then it must be an ellipse: this is Kepler's second law.