Outline

Introduction

Why $E(2n, k)$ is a Rational Times π^{2n}

Guessing the General Theorem

Symmetric Functions

Proving the General Theorem

A Bernoulli-Number Formula

Multiple Zeta Values of Even Arguments

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Outline

1 Introduction

2 Why $E(2n, k)$ is a Rational Times π^{2n}

3 Guessing the General Theorem

4 Symmetric Functions

5 Proving the General Theorem

6 A Bernoulli-Number Formula

Multiple Zeta Values

Multiple Zeta Values of Even Arguments
ME Hoffman
Outline
Introduction
Why $E(2n,k)$ is a Rational Times π^{2n}
Guessing the General Theorem
Symmetric Functions
Proving the General Theorem
A Bernoulli-Number Formula

The multiple zeta values (MZVs) are defined by

$$\zeta(i_1,\ldots,i_k) = \sum_{n_1>\cdots>n_k\geq 1} \frac{1}{n_1^{i_1}\cdots n_k^{i_k}},$$

where $i_1,\ldots,i_k$ are positive integers with $i_1>1$. We call k the depth of the MZV, and $i_1+\cdots+i_k$ its weight. There are many remarkable relations among MZVs, starting with $\zeta(2,1)=\zeta(3)$ (which is often rediscovered and posed as a problem, but was known to Euler). One interesting fact is that all known relations are homogeneous by weight, but we are very far from proving this must be true.

Ancient History

Euler studied MZVs of depth 1 (which are just the values of the "Riemann" zeta function) and obtained the famous formula

$$\zeta(2n) = \frac{(-1)^{n-1} B_{2n} (2\pi)^{2n}}{2(2n)!}.$$

Somewhat less famously, he also studied MZVs of depth 2, and of the many formulas he gave two are of interest here. First, he found that the sum of all the depth-2 MZVs of weight m is just $\zeta(m)$, that is

$$\sum_{i=2}^{m-1} \zeta(i, m-i) = \zeta(m).$$

Second, he gave an alternating sum formula for even weights:

$$\sum_{i=2}^{2n-1} (-1)^i \zeta(i, 2n-i) = \frac{1}{2}\zeta(2n).$$

Recent History

When MZVs of arbitrary depth started to be studied around 1988, one of the first results to attract interest was the generalization of Euler's sum theorem

$$\sum_{i_1,\ldots,i_k=m,\ i_1>1} \zeta(i_1,\ldots,i_k) = \zeta(m).$$

This was conjectured by C. Moen (who proved it for $k=3$) and proved in general by A. Granville and D. Zagier. More recently there has been interest in the sums of MZVs of fixed weight with even arguments, i.e.,

$$E(2n,k) = \sum_{i_1+\cdots+i_k=n} \zeta(2i_1,\ldots,2i_k).$$

Formulas for Even-Argument Sums

Multiple Zeta Values of Even Arguments
ME Hoffman
Outline
Introduction
Why $E(2n, k)$ is a Rational Times π^{2n}
Guessing the General Theorem
Symmetric Functions
Proving the General Theorem
A Bernoulli-Number Formula

If you add Euler's sum theorem in weight $2n$ to his alternating sum theorem in the same weight, you get

$$E(2n, 2) = \frac{3}{4}\zeta(2n).$$

Now it is true that $E(2n, k)$ is always a rational multiple of $\zeta(2n)$ (as I will shortly explain), but for $k > 2$ this multiple depends on the weight $2n$. Nevertheless, it is possible to get formulas with weight-free coefficients if extra terms are allowed.

The Shen-Cai Formulas

Earlier this year Z. Shen and T. Cai published the following formulas for even-argument sums of depth 3 and 4:

$$E(2n,3) = \frac{5}{8}\zeta(2n) - \frac{1}{4}\zeta(2)\zeta(2n-2)$$
$$E(2n,4) = \frac{35}{64}\zeta(2n) - \frac{5}{16}\zeta(2)\zeta(2n-2).$$

What caught my eye was the coefficient of $\zeta(2n)$. Along with $E(2n,1) = \zeta(2n)$ and $E(2n,2) = \frac{3}{4}\zeta(2n)$ we have a sequence

$$1, \quad \frac{3}{4}, \quad \frac{5}{8}, \quad \frac{35}{64}, \ \ldots$$

The Shen-Cai Formulas

Multiple Zeta Values of Even Arguments
ME Hoffman
Outline
Introduction
Why $E(2n,k)$ is a Rational Times π^{2n}
Guessing the General Theorem
Symmetric Functions
Proving the General Theorem
A Bernoulli-Number Formula

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$$1, \quad \frac{3}{4}, \quad \frac{5}{8}, \quad \frac{35}{64}, \ \ldots$$

$$1, \quad 1\cdot\frac{3}{4}, \quad 1\cdot\frac{3}{4}\cdot\frac{5}{6}, \quad 1\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}, \ \ldots$$

Symmetric Sums

Multiple Zeta Values of Even Arguments
ME Hoffman
Outline
Introduction
Why $E(2n,k)$ is a Rational Times π^{2n}
Guessing the General Theorem
Symmetric Functions
Proving the General Theorem
A Bernoulli-Number Formula

Now I'll explain why $E(2n,k) \in \mathbb{Q}\pi^{2n}$.

Theorem (Hoffman, 1992)

Let $i_1, \ldots, i_k$ be integers larger than 1. If $I = (i_1, \ldots, i_k)$ and the symmetric group S_k acts on I in the obvious way, then

$$\sum_{\sigma \in S_k} \zeta(\sigma \cdot I) = \sum_{\text{partitions } \Pi \text{ of } \{1,\ldots,k\}} c(\Pi)\zeta(I,\Pi),$$

where for the partition $\Pi = \{P_1, \ldots, P_t\}$ we define

$$c(\Pi) = (-1)^{k-t}(|P_1|-1)! \cdots (|P_t|-1)!$$

$$\text{and} \quad \zeta(I,\Pi) = \prod_{s=1}^{t} \zeta\left(\sum_{j \in P_s} i_j\right).$$

Symmetric Sums Cont'd

This theorem shows that any symmetric sum of MZVs (i.e., invariant under permutation of the exponent string) is a rational linear combination of products of ordinary MZVs. Thus $\zeta(6,2)+\zeta(2,6)=\zeta(6)\zeta(2)-\zeta(8)$, but it is unknown if $\zeta(6,2)$ is such a linear combination. In my 1992 paper I used the theorem to prove that

$$E(2n,n)=\zeta(\underbrace{2,2,\ldots,2}_{n})=\frac{\pi^{2n}}{(2n+1)!}.$$

Clearly any $E(2n,k)$ is a symmetric sum, and although the details look messy it must be linear combination of products of even zetas homogeneous by weight, and so a rational multiple of π^{2n} (hence of $\zeta(2n)$).

But *Which* Rational Multiple?

The problem with using my theorem to actually compute things is that it's a sum over *set* partitions, which increase according to Bell numbers

$$1,\ 2,\ 5,\ 15,\ 52,\ 203,\ 877,\ 4140\ldots$$

Further, the $E(2n,k)$ get to be more complicated symmetric sums as k gets smaller than n:

$$E(2n,n-1) = \zeta(4,2,\ldots,2)+\zeta(2,4,2,\ldots,2)+\cdots+\zeta(2,\ldots,2,4)$$

$$E(2n,n-2) = \zeta(4,4,2,\ldots,2)+\cdots+\zeta(2,2,\ldots,2,4,4)+ \\ \zeta(6,2,\ldots,2)+\cdots+\zeta(2,\ldots,2,6)$$

But *Which* Rational Multiple? Cont'd

For example, it takes a fairly heroic calculation to get $E(14,5)$ directly from my theorem: one has to add two sums of 52 products each, which eventually reduces to

$$\frac{1}{24}\zeta(2)^4\zeta(6)+\frac{1}{12}\zeta(2)^3\zeta(4)^2-\frac{3}{4}\zeta(2)^2\zeta(4)\zeta(6)-\frac{1}{4}\zeta(2)^3\zeta(8)$$
$$-\frac{1}{4}\zeta(2)\zeta(4)^3+\frac{7}{4}\zeta(2)\zeta(4)\zeta(8)+\frac{19}{24}\zeta(4)^2\zeta(6)+\frac{5}{6}\zeta(2)\zeta(6)^2$$
$$+\zeta(2)^2\zeta(10)-\frac{7}{4}\zeta(6)\zeta(8)-2\zeta(4)\zeta(10)-\frac{5}{2}\zeta(2)\zeta(12)+3\zeta(14)$$
$$=\frac{\pi^{14}}{29189160000}.$$

Fortunately I didn't persist in this foolishness for too long.

But *Which* Rational Multiple? Cont'd

Multiple Zeta Values of Even Arguments
ME Hoffman
Outline
Introduction
Why $E(2n,k)$ is a Rational Times π^{2n}
Guessing the General Theorem
Symmetric Functions
Proving the General Theorem
A Bernoulli-Number Formula

I realized that

$$E(2k,k) = \frac{\pi^{2k}}{(2k+1)!}$$

has some only slightly more complicated cousins, e.g.,

$$E(2k+2,k) = \frac{k\pi^{2k+2}}{3(2k+1)!(2k+3)}$$

$$E(2k+4,k) = \frac{k(7k+13)\pi^{2k+4}}{90(2k+1)!(2n+3)(2n+5)},$$

the latter of which makes the calculation of $E(14,5)$ *much* faster. I'll say where these formulas came from later, but first I'll explain what I did with them.

Guessing the General Theorem

My idea was to come up with extensions of the Shen-Cai formulas. Following the pattern of the leading coefficients, I decided the formula for $k = 5$ should look like

$$E(2n,5) = \frac{63}{128}\zeta(2n) + a\zeta(2)\zeta(n-2) + b\zeta(4)\zeta(n-4)$$

for constants a and b. Setting $n = 5$ and $n = 6$ gave me two equations in two unknowns, which I solved to get $a = -\frac{21}{64}$ and $b = \frac{3}{64}$. So my guess was now

$$E(2n,5) = \frac{63}{128}\zeta(2n) - \frac{21}{64}\zeta(2)\zeta(n-2) + \frac{3}{64}\zeta(4)\zeta(n-4),$$

which I tested at $n = 7$: this is why I wanted to know $E(14,5)$. The formula passed the test, so I worked out a conjectural formula for $E(2n,6)$ in a similar way.

Accumulating Data

My idea was that the general formula looked like

$$E(2n,k) = \frac{1}{2^{2(k-1)}}\binom{2k-1}{k}\zeta(2n) + \sum_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} c_{k,j}\zeta(2j)\zeta(2n-2j),$$

and by solving systems with k close to n I got the following coefficients $c_{k,j}$ (starting with $k=3$)

$$\begin{array}{llll}
k=3: & c_{3,1} = -\frac{1}{4} & & \\
k=4: & c_{4,1} = -\frac{5}{16} & & \\
k=5: & c_{5,1} = -\frac{21}{64} & c_{5,2} = \frac{3}{64} & \\
k=6: & c_{6,1} = -\frac{21}{64} & c_{6,2} = \frac{21}{256} & \\
k=7: & c_{7,1} = -\frac{165}{512} & c_{7,2} = \frac{27}{256} & c_{7,3} = -\frac{3}{1024} \\
k=8: & c_{8,1} = -\frac{1287}{4096} & c_{8,2} = \frac{495}{4096} & c_{8,3} = -\frac{27}{4096}
\end{array}$$

Guessing Coefficients

All these coefficients are consistent with the formulas

$$c_{k,1} = \frac{-1}{2^{2(k-2)}} \binom{2k-3}{k}$$

$$c_{k,2} = \frac{3}{2^{2(k-2)}} \binom{2k-5}{k}$$

$$c_{k,3} = \frac{-3}{2^{2(k-2)}} \binom{2k-7}{k}$$

so I decided that

$$c_{k,j} = \frac{a_j}{2^{2(k-2)}} \binom{2k-2j-1}{k}$$

for some sequence $a_1, a_2, \ldots$.

What is the Next Term of this Sequence?

Multiple Zeta Values of Even Arguments

ME Hoffman

Outline

Introduction

Why $E(2n, k)$ is a Rational Times π^{2n}

Guessing the General Theorem

Symmetric Functions

Proving the General Theorem

A Bernoulli-Number Formula

Now I pressed ahead, just solving for a_j by using the equation $E(2k, k) = \pi^{2k}/(2k+1)!$ repeatedly. So my sequence was

$$a_1 = -1, \ a_2 = 3, \ a_3 = -3,$$

What is the Next Term of this Sequence?

Multiple Zeta Values of Even Arguments

ME Hoffman

Outline

Introduction

Why $E(2n, k)$ is a Rational Times π^{2n}

Guessing the General Theorem

Symmetric Functions

Proving the General Theorem

A Bernoulli-Number Formula

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What is the Next Term of this Sequence?

Multiple Zeta Values of Even Arguments
ME Hoffman
Outline
Introduction
Why $E(2n, k)$ is a Rational Times π^{2n}
Guessing the General Theorem
Symmetric Functions
Proving the General Theorem
A Bernoulli-Number Formula

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What is the Next Term of this Sequence?

Multiple Zeta Values of Even Arguments

ME Hoffman

Outline

Introduction

Why $E(2n, k)$ is a Rational Times π^{2n}

Guessing the General Theorem

Symmetric Functions

Proving the General Theorem

A Bernoulli-Number Formula

Now I pressed ahead, just solving for a_j by using the equation $E(2k, k) = \pi^{2k}/(2k+1)!$ repeatedly. So my sequence was

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What is the Next Term of this Sequence?

Multiple Zeta Values of Even Arguments
ME Hoffman
Outline
Introduction
Why $E(2n,k)$ is a Rational Times π^{2n}
Guessing the General Theorem
Symmetric Functions
Proving the General Theorem
A Bernoulli-Number Formula

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It did not escape my notice that 691 is the numerator of the Bernoulli number B_{12}. In fact all the data was consistent with

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It did not escape my notice that 691 is the numerator of the Bernoulli number B_{12}. In fact all the data was consistent with

$$a_j = \frac{-1}{(4j+2)B_{2j}}$$

and so

$$c_{k,j} = \frac{-1}{2^{2k-3}(2j+1)B_{2j}} \binom{2k-2j-1}{k}.$$

Symmetric Functions

Now I had my formula; I just had to prove it. In fact, the proof involved a detour through the algebra Sym of symmetric functions. Here are some definitions.

Let $x_1, x_2, \ldots$ all have degree 1, and let $\mathfrak{P} \subset \mathbb{Q}[[x_1, x_2, \ldots]]$ be the set of formal power series in the x_i of bounded degree. Then $\mathfrak{P}$ is a graded $\mathbb{Q}$-algebra. An element $f \in \mathfrak{P}$ is symmetric if the coefficient of any term

$$x_{i_1}^{n_1} \cdots x_{i_k}^{n_k}$$

with $i_1, \ldots, i_k$ distinct, agrees with that of

$$x_1^{n_1} \cdots x_k^{n_k}.$$

The set of such f forms an algebra Sym $\subset \mathfrak{P}$.

Bases for Sym

For any integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, the **monomial** symmetric function m_λ is the "smallest" symmetric function containing the monomomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$. For example,

$$m_{21} = x_1^2 x_2 + x_1^2 x_3 + \cdots + x_2^2 x_3 + \cdots + x_1 x_2^2 + x_1 x_3^2 + \cdots$$

The set $\{m_\lambda | \lambda \text{ is a partition}\}$ is an integral basis for Sym.
The **elementary** symmetric functions are

$$e_k = m_{(1^k)},$$

where (1^k) is the partition of k into 1's.
The **complete** symmetric functions are

$$h_k = \sum_{\lambda \text{ is a partition of } k} m_\lambda.$$

Bases for Sym Cont'd

The **power-sum** symmetric functions are

$$p_k = m_{(k)}.$$

If for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ we let

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$$

and similarly h_λ and p_λ, then it is well known that

$$\{e_\lambda | \lambda \text{ is a partition}\} \text{ and } \{h_\lambda | \lambda \text{ is a partition}\}$$

are integral bases for Sym, while

$$\{p_\lambda | \lambda \text{ is a partition}\}$$

is a rational basis.

Generating Functions

The generating function of the **elementary** symmetric functions is

$$E(t) = \sum_{i \geq 0} e_i t^i = \prod_{i \geq 1} (1 + t x_i),$$

while that for the **complete** symmetric functions is

$$H(t) = \sum_{i \geq 0} h_i t^i = \prod_{i \geq 1} \frac{1}{1 - t x_i} = E(-t)^{-1}.$$

The logarithmic derivative of the latter is the generating function $P(t)$ for the **power sums**:

$$P(t) = \frac{d}{dt} \log H(t) = -\sum_{i \geq 1} \frac{d}{dt} \log(1 - t x_i) = \sum_{i \geq 1} p_i t^{i-1}.$$

Even MZVs as Homomorphic Images

Multiple Zeta Values of Even Arguments

ME Hoffman

Outline

Introduction

Why $E(2n, k)$ is a Rational Times π^{2n}

Guessing the General Theorem

Symmetric Functions

Proving the General Theorem

A Bernoulli-Number Formula

There is a homomorphism $\mathfrak{Z} : \mathrm{Sym} \to \mathbb{R}$ induced by sending x_i to $1/i^2$. Then

$$\mathfrak{Z}(p_k) = \zeta(2k), \quad \mathfrak{Z}(e_k) = E(2k, k) = \frac{\pi^{2k}}{(2k+1)!}$$

and

$$\mathfrak{Z}(h_k) = \sum_{i=1}^{k} E(2k, i).$$

For $k \leq n$, we define symmetric functions $N_{n,k}$ so that $\mathfrak{Z}(N_{n,k}) = E(2n, k)$: that is, $N_{n,k}$ is the sum of all monomial symmetric functions corresponding to partitions of n having length k.

Recurrences

Now one of the first things I proved about the $N_{n,k}$ was the recurrence

$$p_1 N_{n-1,k} + p_2 N_{n-2,k} + \cdots + p_{n-k} N_{k,k} = (n-k) N_{n,k} + (k+1) N_{n,k+1},$$

which somewhat resembles the standard recurrence

$$p_1 h_{n-1} + p_2 h_{n-2} + \cdots + p_{n-1} h_1 + p_n = n h_n$$

of power-sum and complete symmetric functions. In particular,

$$p_1 N_{k,k} = N_{k+1,k} + (k+1) N_{k+1,k+1},$$

or

$$N_{k+1,k} = p_1 N_{k,k} - (k+1) N_{k+1,k+1}.$$

Recurrences Cont'd

Apply $\mathfrak{Z}$ to the latter equation to get

$$E(2k+2,k) = \zeta(2)E(2k,k) - (k+1)E(2k+2,k+1)$$

leading to the formula

$$\begin{aligned} E(2k+2,k) &= \frac{\pi^{2m+2}}{6(2k+1)!} - \frac{(k+1)\pi^{2k+2}}{(2k+3)!} \\ &= \frac{k\pi^{2k+2}}{3(2k+1)!(2k+3)} \end{aligned}$$

we saw earlier; the formula for $E(2k+4,k)$ arises in a similar way. In fact, the desire for such formulas is what led me to prove the recurrence.

Generating Function for the $N_{n,k}$

It's actually easy to write the generating function

$$\mathcal{F}(t,s) = 1 + \sum_{n \geq k \geq 1} N_{n,k} t^n s^k,$$

if we remember that the power of t keeps track of degree, while s must keep track of the number of *distinct* x_i's involved in a symmetric function:

$$\begin{aligned}\mathcal{F}(t,s) &= \prod_{i=1}^{\infty}(1 + stx_i + st^2x_i^2 + st^3x_i^3 + \cdots) \\ &= \prod_{i=1}^{\infty} \frac{1 + (s-1)tx_i}{1 - tx_i} = H(t)E((s-1)t).\end{aligned}$$

Generating Function for the $E(2n, k)$

Multiple Zeta Values of Even Arguments

ME Hoffman

Outline

Introduction

Why $E(2n, k)$ is a Rational Times π^{2n}

Guessing the General Theorem

Symmetric Functions

Proving the General Theorem

A Bernoulli-Number Formula

Now let's apply $\mathfrak{Z}$ to this result: if

$$F(t, s) = 1 + \sum_{n \geq k \geq 1} E(2n, k) t^n s^k$$

is the generating function for the $E(2n, k)$, we have

$$F(t, s) = \mathfrak{Z}(\mathcal{F}(t, s)) = \mathfrak{Z}(H(t))\mathfrak{Z}(E((s-1)t)$$

From our formula for $\zeta(e_i)$ we have

$$\mathfrak{Z}(E(t)) = \sum_{i=0}^{\infty} \frac{\pi^{2i} t^i}{(2i+1)!} = \frac{\sinh(\pi\sqrt{t})}{\pi\sqrt{t}}.$$

Generating Function for the $E(2n, k)$ Cont'd

But $H(t) = E(-t)^{-1}$, so

$$\mathfrak{Z}(H(t)) = \frac{\pi\sqrt{-t}}{\sinh \pi\sqrt{-t}} = \frac{\pi\sqrt{t}}{\sin \pi\sqrt{t}}$$

and thus

$$\begin{aligned} F(t,s) =& \mathfrak{Z}(H(t))\mathfrak{Z}(E(-(1-s)t)) \\ =& \frac{\pi\sqrt{t}}{\sin \pi\sqrt{t}} \frac{\sin(\pi\sqrt{1-s}\sqrt{t})}{\pi\sqrt{1-s}\sqrt{t}} \\ =& \frac{\sin(\pi\sqrt{t}\sqrt{1-s})}{\sqrt{1-s}\sin \pi\sqrt{t}}. \end{aligned}$$

Using the Generating Function

Now that we have the generating function, we can use any computer algebra system that does series (like PARI or Maple) to spit out as many of the $E(2n, k)$ as we like. This makes it easy to compute not only $E(14, 5)$, but also, say,

$$E(28, 9) = \frac{19697\pi^{28}}{1423274702804081487360000000}.$$

(In practice one finds that the hardest thing is finding the right term in the output and reading out all the digits correctly.)

Expanding out the Generating Function

There's still the problem of proving the conjecture. We want a formula for $E(2n,k)$ for fixed k, which means we want to look at the coefficient of s^k in $F(t,s)$. Writing

$$F(t,s) = 1 + \sum_{k\geq 1} s^k G_k(t),$$

then it is easy to see from the explicit formula that

$$\begin{aligned} G_k(t) =& (-1)^k \frac{\pi\sqrt{t}}{\sin \pi\sqrt{t}} \sum_{j\geq k} \frac{(-1)^j \pi^{2j} t^j}{(2j+1)!} \binom{j}{k} \\ =& (-1)^k \frac{\pi\sqrt{t}}{\sin \pi\sqrt{t}} \frac{(-t)^k}{k!} \frac{d^k}{dt^k} \left(\frac{\sin \pi\sqrt{t}}{\pi\sqrt{t}} \right). \end{aligned}$$

A Little Calculus Problem

Multiple Zeta Values of Even Arguments

ME Hoffman

Outline

Introduction

Why $E(2n, k)$ is a Rational Times π^{2n}

Guessing the General Theorem

Symmetric Functions

Proving the General Theorem

A Bernoulli-Number Formula

So we have the small calculus problem of finding the nth derivative of $\sin(\pi\sqrt{t})/\pi\sqrt{t}$. Evidently

$$\frac{(-t)^n}{n!}\frac{d^n}{dt^n}\left(\frac{\sin\pi\sqrt{t}}{\pi\sqrt{t}}\right) = P_n(\pi^2 t)\cos\pi\sqrt{t} + Q_n(\pi^2 t)\frac{\sin\pi\sqrt{t}}{\pi\sqrt{t}}$$

for some polynomials P_n, Q_n.

A Little Calculus Problem

Multiple Zeta Values of Even Arguments
ME Hoffman
Outline
Introduction
Why $E(2n,k)$ is a Rational Times π^{2n}
Guessing the General Theorem
Symmetric Functions
Proving the General Theorem
A Bernoulli-Number Formula

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for some polynomials P_n, Q_n. One finds that

$$P_n(x) = -\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{(-4x)^j}{2^{2n-1}(2j+1)!}\binom{2n-2j-1}{n}$$

$$Q_n(x) = \sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-4x)^j}{2^{2n}(2j)!}\binom{2n-2j}{n}.$$

Expanding out the Generating Function

Multiple Zeta Values of Even Arguments
ME Hoffman
Outline
Introduction
Why $E(2n,k)$ is a Rational Times π^{2n}
Guessing the General Theorem
Symmetric Functions
Proving the General Theorem
A Bernoulli-Number Formula

Making use of our calculus exercise, we can now write

$$G_k(t) = -\sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-1}(2j+1)!}\binom{2k-2j-1}{k}\pi\sqrt{t}\cot\pi\sqrt{t}$$

$$+\sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k}(2j)!}\binom{2k-2j}{k}$$

$$= \frac{1}{2}(1-\pi\sqrt{t}\cot\pi\sqrt{t})\sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2}(2j+1)!}\binom{2k-2j-1}{k}$$

$$+ \text{terms of degree} < k.$$

Restating the Conjecture

Now going back to the statement of our conjecture, we can write it a bit more simply if we use Euler's formula for $\zeta(2j)$; this cancels the Bernoulli number in the denominator, leaving us with the cleaner version

$$E(2n,k) = \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-1)^j \pi^{2j} \zeta(2n-2j)}{2^{2k-2j-2}(2j+1)} \binom{2k-2j-1}{k}.$$

Thus, we will be done if we can show $G_k(t)$ is equal to

$$\sum_{n \geq k} t^k \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-1)^j \pi^{2j} \zeta(2n-2j)}{2^{2k-2j-2}(2j+1)} \binom{2k-2j-1}{k}.$$

Restating the Conjecture Cont'd

Multiple Zeta Values of Even Arguments
ME Hoffman
Outline
Introduction
Why $E(2n,k)$ is a Rational Times π^{2n}
Guessing the General Theorem
Symmetric Functions
Proving the General Theorem
A Bernoulli-Number Formula

Write the latter sum in the form

$$\sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2}(2j+1)!} \binom{2k-2j-1}{k} \sum_{n \geq k} \zeta(2n-2j) t^{n-j},$$

and note that

$$\begin{aligned}
\sum_{n \geq k} \zeta(2n-2j) t^{n-j} &= \sum_{n \geq j+1} \zeta(2n-2j) t^{n-j} - \sum_{n=j+1}^{k-1} \zeta(2n-2j) t^{n-j} \\
&= \sum_{m \geq 2} \zeta(m) t^m - \sum_{n=j+1}^{k-1} \zeta(2n-2j) t^{n-j} \\
&= \frac{1}{2}(1 - \pi\sqrt{t}\cot \pi\sqrt{t}) - \sum_{n=j+1}^{k-1} \zeta(2n-2j) t^{n-j}.
\end{aligned}$$

Conclusion

Hence the conjecture reads

$$G_k(t) = \frac{1}{2}(1-\pi\sqrt{t}\cot\pi\sqrt{t}) \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2}(2j+1)!} \binom{2k-2j-1}{k} + \text{terms of degree} < k,$$

where the "terms of degree $< k$" are exactly those necessary to cancel the terms of degree $< k$ in the preceeding sum. But this agrees with the result we got by expanding out the generating function, so we're done.

Another Formula for $E(2n, k)$

We can get another sum for $E(2n, k)$ as follows. Using the recurrence mentioned earlier, we can obtain an explicit formula for $N_{n,k}$ in terms of elementary and complete symmetric functions:

$$N_{n,k} = \sum_{i=0}^{n-k} \binom{n-i}{k} (-1)^{n-k-i} h_i e_{n-i}.$$

Apply $\mathfrak{Z}$ to get

$$E(2n, k) = \frac{(-1)^{n-k-1}\pi^{2n}}{(2n+1)!} \sum_{i=0}^{n-k} \binom{n-i}{k} \binom{2n+1}{2i} 2(2^{2i-1}-1)B_{2i}.$$

Notice that this has $n-k+1$ terms rather than $\lfloor \frac{k-1}{2} \rfloor + 1$ terms as the other formula does.

A Bernoulli-Number Formula

Now if we equate our two formulas for $E(2n,k)$ and use Euler's formula to eliminate values of zeta functions in favor of Bernoulli numbers, we get the identity

$$\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{2k-2i-1}{k} \binom{2n+1}{2i+1} B_{2n-2i} =$$

$$(-1)^k 2^{2k-2n} \sum_{i=0}^{n-k} \binom{n-i}{k} \binom{2n+1}{2i} (2^{2i-1}-1) B_{2i}, \quad k \leq n.$$

Curiously, there is a known Bernoulli-number identity that says the left-hand side equals $\frac{2n+1}{2}\binom{2k-2n}{k}$ on the complementary range $k > n$.