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Multiple Zeta Values and Quasi-Symmetric Functions: Theme and Variations

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Séminaire du LaCIM
18 March 2011

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Multiple Zeta Values

For positive integers $i_1, i_2, \ldots i_k$ with $i_1 > 1$ (to ensure convergence), we define the multiple zeta value (henceforth MZV) $\zeta(i_1, i_2 \ldots, i_k)$ as the sum of the k-fold series

$$\sum_{n_1 > n_2 \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}.$$

We call k the length of $\zeta(i_1, \ldots, i_k)$, and $i_1 + \cdots + i_k$ its weight. Of course, the MZVs of length 1 are the values $\zeta(i)$ of the Riemann zeta function at positive integers $i > 1$.
MZVs go back to Euler (in the case $k = 2$), and recently have appeared in a remarkable number of contexts—from the Kontsevich knot invariants to calculations in perturbative quantum field theory. We will mention some topological appearances of MZVs later in this talk.

Rational Reducibility

It is natural to ask if all MZVs can be written in terms of ordinary zeta values. We say an MZV is *rationally reducible* if it is a rational linear combination of products of the $\zeta(i)$. In fact all MZVs through weight 7 are rationally reducible, e.g.,

$$\zeta(2,1,3) = \zeta(3)^2 - \frac{13}{16}\zeta(6).$$

But this seems to hit a brick wall in weight 8: ever since Euler, no one has been able to write $\zeta(6,2)$ as a rational linear combination of products of the $\zeta(i)$. (Euler falsely claimed in a 1743 letter to Goldbach that

$$\zeta(6,2) = 2\zeta(3)\zeta(5) - \frac{3}{2}\zeta(4)^2 - \frac{3}{4}\zeta(8),$$

but this claim is nowhere to be found in his 1776 paper.)

Rational Reducibility Cont'd

By Euler's well-known formula

$$\zeta(2i) = \frac{(2\pi)^{2i}}{2(2i)!}|B_{2i}| \tag{1}$$

all the $\zeta(i)$ for even i are rational multiples of powers of π^2, so a MZV is rationally reducible if it is a rational linear combination of products of π^2 and $\zeta(i)$ for odd i. It is not even known if $\zeta(5)$ is irrational, so it is unrealistic to expect to prove that, say, $\zeta(6,2)$ is not a rational linear combination of products of $\zeta(3)$, $\zeta(5)$, and π^2, but in the other direction some classes of MZVs can be proved rationally reducible. For example, Euler actually showed in his 1776 paper that $\zeta(m,n)$ is rationally reducible if $m+n$ is odd.

Height of MZVs

Call an exponent string $(i_1, \ldots, i_k)$ admissible if $i_1 > 1$, and minimal admissible if it cannot be written as the juxtaposition of shorter admissible substrings. For example, $(3, 1, 1, 2)$ is admissible, but not minimal admissible (since it is the juxtaposition of $(3, 1, 1)$ with (2)). Any admissible string S is a juxtaposition of $h \geq 1$ minimal admissible substrings: we call h the height of S (so, e.g., $(3, 1, 1, 2)$ has height 2).

Rational Reducibility Results

Here are two results that follow from the relation between MZVs and the quasi-symmetric functions we are about to describe.

Theorem (Symmetric Combination)

If $i_1, \ldots, i_k$ are all greater than or equal to 2, the symmetric combination

$$\sum_{\sigma \in S_n} \zeta(i_{\sigma(1)}, \ldots, i_{\sigma(k)}) \tag{2}$$

is a rational linear combination of products of the $\zeta(i)$'s. In particular, any MZV of the form $\zeta(k, k, \ldots, k)$ with $k > 1$ is rationally reducible.

Theorem (Height One)

Any height one MZV is rationally reducible.

Symmetric and Quasi-Symmetric Functions

Let $x_1, x_2, \ldots$ be a countable sequence of indeterminates, each of degree 1, and let

$$\mathfrak{P} \subset \mathbb{Q}[[x_1, x_2, \ldots]]$$

be the set of formal power series in the x_i of bounded degree: $\mathfrak{P}$ is a graded $\mathbb{Q}$-algebra. Any $f \in \mathfrak{P}$ is quasi-symmetric if the coefficients in f of

$$x_{i_1}^{p_1} x_{i_2}^{p_2} \cdots x_{i_n}^{p_n} \quad \text{and} \quad x_{j_1}^{p_1} x_{j_2}^{p_2} \cdots x_{j_n}^{p_n}$$

agree whenever $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$. The quasi-symmetric functions QSym form an algebra, which properly includes the algebra Sym of symmetric functions, e.g.,

$$\sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \cdots \tag{3}$$

is quasi-symmetric but not symmetric.

Monomial (Quasi)symmetric Functions

For a composition (ordered sequence of positive integers) $I = (i_1, \ldots, i_k)$, the corresponding monomial quasi-symmetric function $M_I \in \mathrm{QSym}$ is defined by

$$M_I = \sum_{n_1 < n_2 < \cdots < n_k} x_{n_1}^{i_1} x_{n_2}^{i_2} \cdots x_{n_k}^{i_k}$$

(so (3) above is $M_{(2,1)}$). Evidently $\{M_I | I \text{ is a composition}\}$ is an integral basis for QSym. For any composition I, let $\pi(I)$ be the partition given by forgetting the ordering. For any partition λ, the monomial symmetric function m_λ is the sum of the M_I with $\pi(I) = \lambda$, e.g.,

$$m_{21} = M_{(2,1)} + M_{(1,2)}.$$

The set $\{m_\lambda | \lambda \text{ is a partition}\}$ is an integral basis for Sym.

Other Bases for Sym

1. The **elementary** symmetric functions are

$$e_k = M_{(1)^k} = m_{\pi((1)^k)},$$

where $(1)^k$ is the composition consisting of k 1's. Then $\{e_\lambda | \lambda \text{ is a partition}\}$ is an integral basis for Sym, where $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$ for $\lambda = \pi(\lambda_1, \lambda_2, \dots)$.

2. The **complete** symmetric functions are

$$h_k = \sum_{|I|=k} M_I = \sum_{|\lambda|=k} m_\lambda.$$

Then $\{h_\lambda | \lambda \text{ is a partition}\}$ is a integral basis for Sym.

3. The **power-sum** symmetric functions are

$$p_k = M_{(k)} = m_k.$$

Then $\{p_\lambda | \lambda \text{ is a partition}\}$ is a rational basis for Sym.

Multiplication of Monomial Quasi-Symmetric Functions

Two monomial quasi-symmetric functions M_I and M_J multiply according to a "quasi-shuffle" rule in which the parts of I and J are shuffled and also combined. For example,

$$\begin{aligned} M_{(1)}M_{(1,2)} &= M_{(1,1,2)} + M_{(1,1,2)} + M_{(1,2,1)} + M_{(2,2)} + M_{(1,3)} \\ &= 2M_{(1,1,2)} + M_{(1,2,1)} + M_{(2,2)} + M_{(1,3)}. \end{aligned}$$

In general the number of terms in the product $M_I M_J$ is

$$\sum_{i=0}^{\min\{\ell(I),\ell(J)\}} \binom{\ell(I)+\ell(J)-i}{i, \ell(I)-i, \ell(J)-i}$$

so, e.g., $M_{(1,1)}M_{(2,1)}$ has $\binom{4}{2} + \binom{3}{111} + \binom{2}{2} = 13$ terms:

$$\begin{aligned} M_{(1,1)}M_{(1,2)} &= 3M_{(1,1,1,2)} + 2M_{(1,1,2,1)} + M_{(1,2,1,1)} \\ &+ M_{(1,2,2)} + M_{(2,1,2)} + M_{(2,2,1)} + 2M_{(1,1,3)} + M_{(1,1,3,1)} + M_{(2,3)}. \end{aligned}$$

Algebraic Structure of QSym

Over the rationals, the algebraic structure of QSym can be described as follows. Order compositions lexicographically:

$$(1) < (1,1) < (1,1,1) < \cdots < (1,2) < \cdots$$
$$< (2) < (2,1) < \cdots < (3) < \cdots$$

A composition I is Lyndon if $I < K$ whenever $I = JK$ for nonempty compositions J, K.

Theorem (Reutenauer-Malevenuto, 1995)

QSym *is a polynomial algebra on* M_I, I *Lyndon.*

Since the only Lyndon composition ending in 1 is (1) itself, the subspace QSym^0 of QSym generated by the M_I such that I ends in an integer greater than 1 is a subalgebra, and in fact $\mathrm{QSym} = \mathrm{QSym}^0[M_{(1)}]$.

The Function $A_- : \mathrm{QSym} \to \mathrm{QSym}$

Consider the linear function from QSym to itself defined by

$$A_-(M_{(a_1,\ldots,a_{k-1},a_k)}) = \begin{cases} M_{(a_1,\ldots,a_{k-1})}, & \text{if } a_k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(Here $M_\emptyset$ is interpreted as 1, so $A_-(M_{(1)}) = 1$.)
From the quasi-shuffle description of multiplication in QSym the following result can be proved.

Proposition

$A_- : \mathrm{QSym} \to \mathrm{QSym}$ *is a derivation.*

Note that $\ker A_- = \mathrm{QSym}^0$. So if we think of QSym as $\mathrm{QSym}^0[M_{(1)}]$, then A_- is differentiation by $M_{(1)}$.

MZVs as Homomorphic Images

There is a homomorphism $\zeta : \mathrm{QSym}^0 \to \mathbb{R}$ given by

$$\zeta(M_I) = \zeta(i_k, i_{k-1}, \ldots, i_1)$$

for $I = (i_1, \ldots, i_k)$, induced by sending $x_j \to \frac{1}{j}$. (This is well-defined since $i_k > 1$ for $M_I \in \mathrm{QSym}^0$.) Note $\zeta(p_i) = \zeta(M_{(i)}) = \zeta(i)$ for $i > 1$. The intersection

$$\mathrm{Sym}^0 = \mathrm{QSym}^0 \cap \mathrm{Sym}$$

is the subspace of Sym generated by the m_λ such that all parts of λ are 2 or more, and is in fact the subalgebra of Sym generated by the p_i with $i > 1$. This proves the Symmetric Combination Theorem: any combination (2) of MZVs is the image under ζ of an element of Sym^0, hence a polynomial in $\zeta(2), \zeta(3), \ldots$ with rational coefficients.

Explicit Formulas

In the case of $\zeta((k)^n)$, $k > 1$, one can be quite explicit: from the theory of symmetric functions

$$e_n = \frac{1}{n!} \begin{vmatrix} p_1 & 1 & 0 & \cdots & 0 \\ p_2 & p_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_{n-2} & p_{n-3} & \cdots & n-1 \\ p_n & p_{n-1} & p_{n-2} & \cdots & p_1 \end{vmatrix}$$

and hence $\zeta((k)^n)$ is

$$\frac{1}{n!} \begin{vmatrix} \zeta(k) & 1 & 0 & \cdots & 0 \\ \zeta(2k) & \zeta(k) & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \zeta((n-1)k) & \zeta((n-2)k) & \zeta((n-3)k) & \cdots & n-1 \\ \zeta(nk) & \zeta((n-1)k) & \zeta((n-2)k) & \cdots & \zeta(k) \end{vmatrix}.$$

Extending ζ to $\zeta_u : \mathrm{QSym} \to \mathbb{R}[u]$

Since $\mathrm{QSym} = \mathrm{QSym}^0[M_{(1)}]$, we can extend ζ to a homomorphism $\zeta_u : \mathrm{QSym} \to \mathbb{R}[u]$ by defining $\zeta_u(w) = \zeta(w)$ for $w \in \mathrm{QSym}^0$ and $\zeta_u(M_{(1)}) = u$. In view of the Proposition above, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{QSym} \arrow[r, "\zeta_u"] \arrow[d, "A_-"'] & \mathbb{R}[u] \arrow[d, "\frac{d}{du}"'] \\
\mathrm{QSym} \arrow[r, "\zeta_u"] & \mathbb{R}[u]
\end{tikzcd}$$

In the case $u = \gamma \approx 0.5572$ (Euler's constant), the homomorphism ζ_u turns out to be especially useful. To explain why we must digress a bit to introduce some generating functions.

Generating Functions

1. The generating function of the **elementary** symmetric functions is

$$E(t) = \sum_{i \geq 0} e_i t^i = \prod_{i \geq 1} (1 + t x_i).$$

2. That for the **complete** symmetric functions is

$$H(t) = \sum_{i \geq 0} h_i t^i = \prod_{i \geq 1} \frac{1}{1 - t x_i} = E(-t)^{-1}.$$

3. The logarithmic derivative of $H(t)$ is the generating function $P(t)$ for the **power sums**:

$$P(t) = \frac{d}{dt} \log H(t) = -\sum_{i \geq 1} \frac{d}{dt} \log(1 - t x_i) = \sum_{i \geq 1} p_i t^{i-1}.$$

Generating Functions Cont'd

The power series expansion of $\psi(1-t)$, where ψ is the logarithmic derivative of the gamma function, is

$$\psi(1-t) = -\gamma - \sum_{i \geq 2} \zeta(i) t^{i-1}.$$

If $u = \gamma$, then $\zeta_u(p_1) = \gamma$ and so $\zeta_u(P(t)) = -\psi(1-t)$. Hence (since $P(t) = \frac{d}{dt} \log H(t)$) we have

Theorem (Generating Function)

$\zeta_\gamma(H(t)) = \Gamma(1-t)$.

From $E(t) = H(-t)^{-1}$ follows $\zeta_\gamma(E(t)) = \Gamma(1+t)^{-1}$.

Generating Functions Cont'd

Now it can be shown analytically that the generating function

$$F(s,t) = \sum_{n,m \geq 1} \zeta(n+1,(1)^{m-1}) s^n t^m$$

of the height one MZVs can be written

$$F(s,t) = 1 - \frac{\Gamma(1-t)\Gamma(1-s)}{\Gamma(1-t-s)}.$$

Applying the Generating Function Theorem, this is

$$F(s,t) = \zeta_\gamma \left(1 - \frac{H(t)H(s)}{H(t+s)}\right).$$

Now

$$H(t) = \exp\left(\sum_{i \geq 1} \frac{p_i t^i}{i}\right) = e^{p_1 t} \exp\left(\sum_{i \geq 2} \frac{p_i t^i}{i}\right),$$

Generating Functions Cont'd

so it follows that

$$1 - \frac{H(t)H(s)}{H(t+s)} = 1 - \exp\left(\sum_{i\geq 2} p_i \frac{t^i + s^i - (t+s)^i}{i}\right)$$

and hence, applying ζ_γ,

$$\sum_{n,m\geq 1} \zeta(n+1,(1)^{m-1}) s^n t^m = 1 - \exp\left(\sum_{i\geq 2} \zeta(i) \frac{t^i + s^i - (t+s)^i}{i}\right).$$

This proves the Height One Theorem: any height-one MZV $\zeta(n+1,(1)^{m-1})$ is a polynomial in the $\zeta(i)$, $i \geq 2$, with rational coefficients, though $M_{((1)^{m-1},n+1)} \notin \mathrm{Sym}$ for $m > 1$.

Hirzebruch's Theory of Genera

A genus is a function ϕ that assigns to any manifold M in some class (say those with an almost complex structure) a number $\phi(M)$, subject to the requirement that

$$\phi(M \times N) = \phi(M)\phi(N).$$

For an almost complex M of dimension $2n$, a genus can be given by

$$\phi(M) = \langle K_n(c_1, \ldots, c_n), [M] \rangle,$$

where the $c_i \in H^{2i}(M; \mathbb{Z})$ are the Chern classes of M, $[M] \in H_{2n}(M; \mathbb{Z})$ is the fundamental class of M, and $\{K_i\}$ is a multiplicative sequence: i.e., a sequence of homogeneous polynomials (with $\deg K_i = i$) so that, if

$$K(1 + c_1 + c_2 + \cdots) = 1 + K_1(c_1) + K_2(c_1, c_2) + \cdots ,$$

then $K(a)K(b) = K(ab)$ for any formal series a, b having constant term 1.

Multiplicative Sequences

In fact, a multiplicative sequence $\{K_i\}$ is determined by the power series

$$K(1+t) = 1 + r_1 t + r_2 t^2 + r_3 t^3 + \ldots, \tag{4}$$

since (by the algebraic independence of the elementary symmetric functions e_i) we can think of the homogeneous terms of an arbitrary formal series as the e_i of new variables $x_1, x_2, \ldots$:

$$1 + e_1 + e_2 + e_3 \cdots = (1+x_1)(1+x_2)\cdots$$

and thus

$$K(1 + e_1 + e_2 + \cdots) = K(1+x_1)K(1+x_2)\cdots .$$

We say $\{K_i\}$ is the multiplicative sequence belonging to the power series (4).

Mirror Symmetry and the Γ-Genus

The Γ-genus is given by

$$\Gamma(M^{2n}) = \langle Q_n(c_1, \ldots, c_n), [M^{2n}] \rangle,$$

where $\{Q_i\}$ is the multiplicative sequence belonging to the series

$$\Gamma(1+t)^{-1} = 1 + \gamma t + \frac{1}{2}(\gamma^2 - \zeta(2))t^2 + \cdots .$$

The polynomials Q_i occur in the context of mirror symmetry, which involves Calabi-Yau manifolds (for our purposes, Kähler manifolds whose first Betti number and first Chern class both vanish).

Mirror Symmetry and the Γ-Genus Cont'd

String theorists are interested in Calabi-Yau manifolds of (real) dimension 6, since they occur in supersymmetry theory. (10-dimensional spacetime looks locally like $\mathbb{R}^4 \times M^6$, where M^6 is a Calabi-Yau manifold.) They found that Calabi-Yau manifolds seemed to come in "mirror pairs," i.e. distinct manifolds that gave the same physics. At present there are mathematically rigorous constructions of mirrors only for certain classes of Calabi-Yau manifolds (or orbifolds), e.g., hypersurfaces or complete intersections in toric varieties.

Mirror Symmetry Cont'd

For one such class, Hosono, Klemm, Theisen and Yau (1995) found relations between the Chern classes of a manifold and period functions on its mirror: e.g., for a Calabi-Yau 3-fold X that is a complete intersection in a product of weighted projective spaces,

$$\langle c_3, [X] \rangle = \frac{1}{6\zeta(3)} K_{ijk} \frac{\partial^3 c}{\partial \rho_i \partial \rho_j \partial \rho_k}(0, \dots, 0). \tag{5}$$

Here $c(\rho_1, \rho_2, \dots)$ are coefficients of a generalized hypergeometric series for the period function on a mirror of X (with gamma functions replacing the factorials so it can be differentiated), and K_{ijk} is the Yukawa coupling (B-model correlation function).

Libgober's Formula

A. Libgober (1999) generalized these relations to higher dimensions: e.g., if X is a Calabi-Yau d-fold which is a hypersurface in a toric Fano manifold satisfying a technical condition,

$$\langle Q_d(c_1, \ldots, c_d), [X] \rangle = \frac{1}{d!} K_{i_1, i_2, \ldots, i_d} \frac{\partial^d c(0, \ldots, 0)}{\partial \rho_{i_1} \partial \rho_{i_2} \cdots \partial \rho_{i_d}} \tag{6}$$

where $K_{i_1, \ldots, i_d}$ is the suitably normalized d-point function corresponding to a mirror $\tilde{X}$ of X and $c(\rho_1, \ldots,)$ are the coefficients of the hypergeometric series for the holomorphic period of $\tilde{X}$ at a maximal degeneracy point (of a partial compactification of the deformation space of $\tilde{X}$).

Libgober's Formula Cont'd

We can write the coefficients of the Q_i in terms of MZVs.

Theorem

The coefficient of e_λ in $Q_i(e_1, \ldots, e_i)$ is $\zeta_\gamma(m_\lambda)$ for any partition λ of i.

Proof.

We use Generating Function Theorem, together with the symmetry of the transition matrices between the bases $\{e_\lambda\}$ and $\{m_\lambda\}$ of Sym:

$$\sum_{j \geq 0} Q_j(e_1, \ldots, e_j) = \prod_{i \geq 1} \frac{1}{\Gamma(1 + x_i)} =$$

$$\prod_{i \geq 1} \sum_{j \geq 0} \zeta_\gamma(e_j) x_i^j = \sum_\lambda \zeta_\gamma(e_\lambda) m_\lambda = \sum_\lambda \zeta_\gamma(m_\lambda) e_\lambda.$$

Libgober's Formula Cont'd

Thus, for example,

$$Q_2(c_1, c_2) = \zeta(2)c_2 + \frac{1}{2}(\gamma^2 - \zeta(2))c_1^2 \tag{7}$$

and

$$Q_3(c_1, c_2, c_3) = \zeta(3)c_3 + (\gamma\zeta(2) - \zeta(3))c_1c_2 + \frac{1}{6}(\gamma^3 - 3\gamma\zeta(2) + 2\zeta(3))c_1^3 \tag{8}$$

In the Calabi-Yau case we have $c_1 = 0$ and terms involving γ don't appear: e.g.,

$$Q_3(0, c_2, c_3) = \zeta(3)c_3$$

which shows that (6) generalizes (5).

Lu's $\hat{\Gamma}$-genus

Rongmin Lu (2008) considered the genus belonging to the power series

$$e^{-\gamma t}\Gamma(1+t)^{-1}, \tag{9}$$

which he calls the $\hat{\Gamma}$-genus, and related it to a regularized S^1-equivariant Euler class.
The logarithmic derivative of (9) is

$$-\gamma - \psi(1+t) = \sum_{i \geq 2} \zeta(i)(-t)^{i-1},$$

which is just the corresponding one for the Γ-genus with γ removed. If we write $\{P_n\}$ for the multiplicative sequence corresponding to the $\hat{\Gamma}$-genus, then P_n is Q_n with γ set to zero.

Lu's Genus Cont'd

That is, the coefficient of e_λ in $P_n(e_1, \ldots, e_n)$ is $\zeta_0(m_\lambda)$. For example,

$$P_2(c_1, c_2) = \zeta(2)c_2 - \frac{1}{2}\zeta(2)c_1^2$$

from equation (7), and

$$P_3(c_1, c_2, c_3) = \zeta(3)c_3 - \zeta(3)c_1c_2 + \frac{1}{3}\zeta(3)c_1^3$$

from equation (8).

Lu's Genus Cont'd

Let $\pi : E \to M$ be an m-dimensional complex vector bundle with a spin structure, $\pi_\ell : LE \to LM$ the corresponding loop bundle (LM is the infinite-dimensional manifold of unbased loops on M), and $\nu(E)$ the E-normal bundle over M (i.e., $i^*(LE)/E$). Lu defined a regularized Euler class $e_{\mathrm{reg}}(\nu(E))$ and gave a formula for it as follows. Let $E = L_1 \oplus L_2 \oplus \cdots \oplus L_m$ be the formal factorization of E into line bundles, $x_i = c_1(L_i)$.

Lu's Genus Cont'd

Then

$$e_{\mathrm{reg}}(\nu(E)) = \left(\frac{2\pi}{\sqrt{v}}\right)^m \prod_{j=1}^{m} \left[f\left(\frac{2\pi x_j}{v}\right)\right]^{-1}, \tag{10}$$

where v is an indeterminate of degree 2 and $f(x) = e^{\gamma x}\Gamma(1+x)$. Since

$$\sum_{j=1}^{m} P_j(c_1, \ldots, c_j) = \prod_{i=1}^{m} (f(x_i))^{-1},$$

equation (10) represents e_{reg} as a kind of normalized $\hat{\Gamma}$-genus.

Hopf Algebra Characters

Now QSym is a graded Hopf algebra, with coproduct

$$\Delta(M_I) = \sum_{I=JK} M_J \otimes M_K$$

where the sum is over all decompositions of I as a juxtaposition (including the cases $J = \emptyset$ and $K = \emptyset$). Also, ζ_γ is a real character of the Hopf algebra QSym (that is, an algebra homomorphism $\mathrm{QSym} \to \mathbb{R}$). A character χ of QSym is even if

$$\chi(w) = (-1)^{|w|}\chi(w) \tag{11}$$

for homogeneous elements w of QSym, and odd if

$$\chi(w) = (-1)^{|w|}\chi(S(w)), \tag{12}$$

where S is the antipode of QSym.

The ABS Theorem

Theorem (Aguiar, Bergeron and Sottille, 2006)

Any real character of a Hopf algebra has a unique decomposition in the convolution algebra into an even character times an odd one.

Thus, there is an even character ζ_+ and an odd character ζ_- so that $\zeta_\gamma = \zeta_+\zeta_-$ in the convolution algebra, i.e.,

$$\zeta_\gamma(w) = \sum_{(w)} \zeta_+(w_{(1)})\zeta_-(w_{(2)}) \tag{13}$$

using Sweedler's notation for coproducts:

$$\Delta(w) = \sum_{(w)} w_{(1)} \otimes w_{(2)}.$$

Primitives

The power sums $p_i = M_{(i)}$ are primitive, i.e.

$$\Delta(p_i) = 1 \otimes p_i + p_i \otimes 1,$$

and so

$$\zeta(i) = \zeta(p_i) = \zeta_-(p_i) + \zeta_+(p_i). \tag{14}$$

Equation (11) implies that $\zeta_+(w) = 0$ for any odd-dimensional w, particularly $w = p_i$, i odd. On the other hand, equation (12) gives $\zeta_-(p_i) = 0$ for i even, since $S(p_i) = -p_i$. Together with equation (14), this says

$$\zeta_+(p_i) = \begin{cases} \zeta(i), & i \text{ even}, \\ 0, & i \text{ odd}, \end{cases} \quad \text{and } \zeta_-(p_i) = \begin{cases} 0, & i \text{ even}, \\ \zeta(i), & i > 1 \text{ odd}, \\ \gamma, & i = 1. \end{cases}$$

Even Parts Theorem

In general, computing $\zeta_+(M_I)$ and $\zeta_-(M_I)$ for an arbitrary monomial quasi-symmetric function M_I is difficult, apart from the general fact that

$$\zeta_+(M_I) = 0 \quad \text{if } |I| \text{ is odd.}$$

Nevertheless, there is the following result, which can be proved from the general theory of Aguiar-Bergeron-Sottille together with a specific result of Aguiar-Hsiao (2004).

Theorem

If all parts of I are even, then $\zeta_-(M_I) = 0$.

Computing ζ_+ and ζ_- on Sym

The situation is dramatically different if ζ_γ is restricted to Sym (which is all we need for the Γ- and $\hat{\Gamma}$-genera). The p_i generate Sym, so for every partition λ

$$m_\lambda = P_\lambda(p_1, p_2, p_3, p_4, \dots)$$

for some polynomial P_λ with rational coefficients. Since ζ_+ and ζ_- are homomorphisms,

$$\zeta_-(m_\lambda) = P_\lambda(\gamma, 0, \zeta(3), 0, \dots)$$

and

$$\zeta_+(m_\lambda) = P_\lambda(0, \zeta(2), 0, \zeta(4), \dots).$$

In view of Euler's identity (1), the latter formula means that $\zeta_+(m_\lambda)$ is a rational multiple of $\pi^{|\lambda|}$ when $|\lambda|$ is even (and of course $\zeta_+(m_\lambda) = 0$ when $|\lambda|$ is odd).

Generating Functions Yet Again

Since the involution of Sym that exchanges the e_i and the h_i leaves the odd p_i fixed, it follows that $\zeta_-(e_n) = \zeta_-(h_n)$, i.e.,

$$\zeta_- \left(\frac{H(t)}{E(t)} \right) = 1.$$

On the other hand, $\zeta_+(E(t))$ is an even function, so $\zeta_+(E(t)) = \zeta_+(E(-t)) = \zeta_+(H(t))^{-1}$ and thus

$$\zeta_+ \left(\frac{H(t)}{E(t)} \right) = \zeta_+(H(t))^2.$$

Hence, using $\zeta_+\zeta_- = \zeta_\gamma$ and the Generating Function Theorem,

$$\zeta_+(H(t))^2 = \zeta_+ \left(\frac{H(t)}{E(t)} \right) \zeta_- \left(\frac{H(t)}{E(t)} \right) = \zeta_\gamma \left(\frac{H(t)}{E(t)} \right) = \Gamma(1-t)\Gamma(1+t).$$

Factored Generating Function Theorem

Because of the reflection formula for the gamma function, this is

$$\zeta_+(H(t))^2 = \frac{\pi t}{\sin \pi t}.$$

Thus we have the following result.

Theorem (Factored Generating Function)

The ABS factors of ζ_γ are given by

$$\zeta_+(E(t)) = \zeta_+(H(t))^{-1} = \sqrt{\frac{\sin \pi t}{\pi t}}$$

$$\zeta_-(E(t)) = \zeta_-(H(t)) = \sqrt{\frac{\sin \pi t}{\pi t}}\Gamma(1-t).$$

Explicit Formula for $\zeta_-(e_n)$

There is also the explicit formula for e_n in terms of power sums:

$$e_n = \sum_{i_1+2i_2+\cdots+ni_n=n} \frac{(-1)^n}{i_1!i_2!\cdots i_n!}(-p_1)^{i_1}\cdots\left(-\frac{p_n}{n}\right)^{i_n},$$

which follows from

$$E(t) = H(-t)^{-1} = \exp\left(-\int_0^{-t} P(s)ds\right).$$

If we apply ζ_- to this, the minus signs cancel nicely to give

$$\zeta_-(e_n) = \sum_{i_1+3i_3+5i_5\cdots=n} \frac{\gamma^{i_1}\zeta(3)^{i_3}\zeta(5)^{i_5}\cdots}{i_1!3^{i_3}i_3!5^{i_5}i_5!\cdots}. \tag{15}$$

An Example

To see how to put these results together, we note that

$$\sqrt{\frac{\sin \pi t}{\pi t}} = 1 - \frac{\pi^2 t^2}{12} + \frac{\pi^4 t^4}{1440} - \cdots$$

Then since the e_i are divided powers,

$$\Delta(e_4) = 1 \otimes e_4 + e_1 \otimes e_3 + e_2 \otimes e_2 + e_3 \otimes e_1 + e_4 \otimes 1$$

and from the Factored Generating Function Theorem together with equations (13) and (15) it follows that

$$\begin{aligned}
\zeta_\gamma(e_4) &= \zeta_-(e_4) + \zeta_+(e_2)\zeta_-(e_2) + \zeta_+(e_4) \\
&= \frac{\gamma^4}{4!} + \frac{\gamma\zeta(3)}{3} - \frac{\pi^2}{12} \cdot \frac{\gamma^2}{2} + \frac{\pi^4}{1440} \\
&= \frac{\gamma^4}{24} - \frac{\gamma^2\zeta(2)}{4} + \frac{\gamma\zeta(3)}{3} + \frac{\zeta(4)}{16}.
\end{aligned}$$