Maxwell's Equations for Electricity and Magnetism

1 Electrostatics

According to Coulomb's Law, the force on a charge q' at location $\vec{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ due to a point charge q at the origin is

$$\frac{1}{4\pi\epsilon_0}\frac{qq'\vec{r}}{|\vec{r}|^3} = \frac{qq'}{4\pi\epsilon_0|\vec{r}|^2}\hat{\mathbf{r}}, \tag{1}$$

where $\hat{\mathbf{r}}$ is the unit vector in the direction of $\vec{r}$ and ϵ_0 is a physical constant (the permittivity of vaccum, 8.854×10^{-12} farad/meter). We can rephrase this law by introducing the electric field

$$\vec{E} = \frac{q}{4\pi\epsilon_0|\vec{r}|^2}\hat{\mathbf{r}} \tag{2}$$

due to the point charge q: then the force on any charge q' is simply $q'\vec{E}$.

Now consider a sphere S_R of radius R centered on the origin. We want to compute the total flux of the electric field (2) through S_R, that is,

$$\iint_{S_R} \vec{E}\cdot d\vec{S} = \iint_{S_R} \frac{q}{4\pi\epsilon_0|\vec{r}|^2}\hat{\mathbf{r}}\cdot d\vec{S} = \frac{q}{4\pi\epsilon_0}\iint_{S_R}\frac{\hat{\mathbf{r}}}{R^2}\cdot d\vec{S}. \tag{3}$$

Using the parametrization

$$\vec{r} = R\cos\theta\sin\phi\mathbf{i} + R\sin\theta\sin\phi\mathbf{j} + R\cos\phi\mathbf{k}$$

for S_R, we have

$$\begin{aligned} d\vec{S} = \frac{\partial\vec{r}}{\partial\phi}\times\frac{\partial\vec{r}}{\partial\theta}d\phi d\theta &= (\cos\theta\sin\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\phi\mathbf{k})R^2\sin\phi d\phi d\theta \\ &= \hat{\mathbf{r}}R^2\sin\phi d\phi d\theta \end{aligned}$$

and so the integral (3) becomes

$$\frac{q}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\frac{\hat{\mathbf{r}}}{R^2}\cdot\hat{\mathbf{r}}R^2\sin\phi d\phi d\theta=\frac{q}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\sin\phi d\phi d\theta=$$
$$\frac{q}{4\pi\epsilon_0}4\pi=\frac{q}{\epsilon_0}. \quad (4)$$

Notice that this result is independent of the radius R of the sphere. In fact this calculation can be generalized: if Σ is *any* closed surface containing the origin, then

$$\iint_\Sigma \vec{E}\cdot d\vec{S}=\frac{q}{\epsilon_0}. \quad (5)$$

Now the electric field is additive: that is, the electric field $\vec{E}$ due to two point charges q and q' is the sum of the electric field due to q and the electric field due to q'. It follows that for any collection of point charges, the electric field $\vec{E}$ satisfies the equation

$$\iint_\Sigma \vec{E}\cdot d\vec{S}=\frac{1}{\epsilon_0}(\text{sum of charges contained in }\Sigma) \quad (6)$$

This is the integral form of Gauss's Law. Notice that if Σ surrounds a positive charge, then both sides of equation (6) are positive: if Σ surrounds a negative charge, both sides of (6) are negative. That is, positive charges are sources of the electric field $\vec{E}$, and negative charges are sinks of $\vec{E}$.

Now if we think of charges as being "spread out" rather than concentrated at a point, there is a charge density ρ so that

$$\iiint_D \rho dV=\text{total charge contained in } D$$

Thus, equation (6) becomes

$$\iint_\Sigma \vec{E}\cdot d\vec{S}=\frac{1}{\epsilon_0}\iiint_D \rho dV,$$

where D is the region inside Σ. Using the divergence theorem, this is

$$\iiint_D \nabla\cdot\vec{E}dV=\frac{1}{\epsilon_0}\iiint_D \rho dV, \quad \text{or} \quad \iiint_D (\nabla\cdot\vec{E}-\frac{\rho}{\epsilon_0})dV=0.$$

Since this holds for *any* region D, we have

$$\nabla\cdot\vec{E}=\frac{\rho}{\epsilon_0}, \quad (7)$$

which is the differential form of Gauss's Law.

2 The Magnetic Field

There is also a magnetic field $\vec{B}$. Stationary charges are not affected by the magnetic field, but moving charges are. In fact, a charge q' with velocity vector $\vec{v}$ will feel a magnetic force $q'\vec{v} \times \vec{B}$. Putting this together with the electrical force, the force felt by a charge q' is the so-called Lorentz force

$$q'(\vec{E} + \vec{v} \times \vec{B}). \tag{8}$$

Unlike the electrical field, the magnetic field $\vec{B}$ has no sources or sinks. That is, there are no isolated magnetic "charges" (magnetic monopoles). Instead, magnets always have a north pole and a south pole: if you break a magnet in half, you get two smaller magnets each with its own north and south pole. The lack of sources and sinks can be expressed as

$$\nabla \cdot \vec{B} = 0. \tag{9}$$

It was discovered experimentally that moving electrical charges (current) create a magnetic field. If we consider a small pierce of wire $d\vec{r}$ with a current I flowing through it, the contribution it makes to the magnetic field at the point $\vec{r}$ is given by the Biot-Savart Law:

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{r} \times \vec{r}}{|\vec{r}|^3} = \frac{\mu_0}{4\pi} \frac{I d\vec{r} \times \hat{\mathbf{r}}}{|\vec{r}|^2}, \tag{10}$$

where μ_0 is a physical constant (the permeability of vacuum, $4\pi \times 10^{-7}$ henry/meter). Note the similarity to Coulomb's Law (2): like the electric field, the magnetic field is inversely proportional to the square of the distance.

Now we consider an infinite straight wire carrying current I, and calculate the magnetic field $\vec{B}$ at a point distance R from the wire. Choose coordinates so $\mathbf{i}$ points down the wire, and our point is at position $R\mathbf{j}$. Then equation (10) says that

$$\vec{B} = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \frac{I\mathbf{i}dx \times (-x\mathbf{i} + R\mathbf{j})}{(x^2 + R^2)^{\frac{3}{2}}} = \frac{\mu_0 \mathbf{k}}{4\pi} \int_{-\infty}^{\infty} \frac{RIdx}{(x^2 + R^2)^{\frac{3}{2}}} = \frac{\mu_0 I}{2\pi R}\mathbf{k}.$$

That is, the field strength is $\frac{\mu_0 I}{2\pi R}$, and the direction is perpendicular to the axis of the wire. It follows that if we take a circle C of radius R whose center is on the wire (and whose diameters are perpendicular to the wire), then

$$\int_C \vec{B} \cdot d\vec{r} = \int_C \frac{\mu_0 I}{2\pi R} dr = \frac{\mu_0 I}{2\pi R}(2\pi R) = \mu_0 I. \tag{11}$$

Note that the result does not depend on R (as in equation (4)). As with equation (4), equation (11) generalizes: For *any* path C around the wire,

$$\int_C \vec{B} \cdot d\vec{r} = \mu_0 I = \mu_0(\text{current enclosed by } C) \tag{12}$$

This is Ampere's Law (in integral form). Now Stokes's theorem says that

$$\int_C \vec{B} \cdot d\vec{r} = \iint_S (\nabla \times \vec{B}) \cdot d\vec{S},$$

where S is a surface with boundary curve C. If we introduce a current density $\vec{J}$ so that

$$I = \iint_S \vec{J} \cdot d\vec{S},$$

then equation (12) becomes

$$\iint_S (\nabla \times \vec{B}) \cdot d\vec{S} = \mu_0 \iint_S \vec{J} \cdot d\vec{S}, \quad \text{or} \quad \iint_S (\nabla \times B - \mu_0 J) \cdot d\vec{S} = 0.$$

As S is arbitrary, this implies

$$\nabla \times B = \mu_0 J, \tag{13}$$

the differential form of Ampere's Law.

A changing magnetic field creates an electric field. This Faraday Induction Law was discovered experimentally, and can be stated as follows. Suppose we have a loop C of wire. Then the line integral $\int_C E \cdot d\vec{r}$ is related to the flux of the magnetic field $\vec{B}$ through a surface S having boundary curve C:

$$\int_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{S} \tag{14}$$

(note the presence of the time derivative here!). That is, the circulation of the electric field is minus the rate of change in the magnetic flux. The integral form (14) has a corresponding differential form. Using Stokes's theorem, equation(14) becomes

$$\iint_S (\nabla \times \vec{E}) d\vec{r} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{S},$$

from which we get the differential equation

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \tag{15}$$

3 Maxwell's Equations

If we collect equations (7), (15), (9) and (13), we have

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{16}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{17}$$

$$\nabla \cdot \vec{B} = 0 \tag{18}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \tag{19}$$

Maxwell studied these equations, and realized there was something not quite right about the last one. If we take divergences of both sides of equation (19), we get (since divergence of a curl is zero)

$$0 = \mu_0 \nabla \cdot \vec{J} \tag{20}$$

But experimentally, electrical charge is seen to be conserved. For any closed surface Σ, the rate at which current flows out of Σ is equal to the *decrease* of total charge in the volume D enclosed by Σ. That is,

$$\iint_\Sigma \vec{J} \cdot d\vec{S} = -\frac{d}{dt} \iiint_D \rho dV$$

Using the divergence theorem, this is

$$\iiint_D \left(\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right) dV = 0.$$

Since D is arbitrary, this means

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}, \tag{21}$$

which contradicts equation (20) if the charge density ρ isn't constant. Maxwell reasoned that since a changing magnetic field $\vec{B}$ produces an electric field $\vec{E}$, perhaps a changing electric field $\vec{E}$ also produces a magnetic field $\vec{B}$. That is, he tried adding a term to equation (19):

$$\nabla \times \vec{B} = \mu_0 \vec{J} + K \frac{\partial \vec{E}}{\partial t}. \tag{22}$$

Now take the divergence of both sides of (22) to get

$$0 = \mu_0 \nabla \vec{J} + K \frac{\partial}{\partial t}(\nabla \times \vec{E}) = \mu_0 \nabla J + \frac{K}{\epsilon_0} \frac{\partial \rho}{\partial t},$$

(using equation (17)), or

$$\nabla \vec{J} = -\frac{K}{\epsilon_0 \mu_0} \frac{\partial \rho}{\partial t}. \tag{23}$$

To make equation (23) agree with (21), we must take

$$K = \mu_0 \epsilon_0$$

and so the corrected equation (19) reads

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t},$$

and the correct Maxwell equations are

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}. \end{aligned}$$

4 A Consequence: Electromagnetic Waves

In a region of space without current or charge, the Maxwell equations imply

$$\nabla \cdot \vec{E} = 0 \tag{24}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{25}$$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}. \tag{26}$$

Now for *any* vector field $\vec{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, it's true that

$$\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - (\nabla^2 F_1 \mathbf{i} + \nabla^2 F_2 \mathbf{j} + \nabla^2 F_3 \mathbf{k}) \tag{27}$$

where $\nabla^2 f$ is the Laplacian

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2},$$

as can be shown by a direct (but somewhat tedious) calculation. Applying equation (27) to $\vec{F} = \vec{E}$ and using equation (24), we have

$$\nabla \times (\nabla \times \vec{E}) = -(\nabla^2 E_1 \mathbf{i} + \nabla^2 E_2 \mathbf{j} + \nabla^2 E_3 \mathbf{k}). \tag{28}$$

On the other hand, using equations (25) and (26), we have

$$\nabla \times (\nabla \times \vec{E}) = -\nabla \times \left(\frac{\partial \vec{B}}{\partial t} \right) = -\frac{\partial}{\partial t}(\nabla \times \vec{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}. \tag{29}$$

Comparing equations (28) and (29), we must have

$$\nabla^2 E_1 = \mu_0 \epsilon_0 \frac{\partial^2 E_1}{\partial t^2}, \quad \nabla^2 E_2 = \mu_0 \epsilon_0 \frac{\partial^2 E_2}{\partial t^2}, \quad \text{and} \quad \nabla^2 E_3 = \mu_0 \epsilon_0 \frac{\partial^2 E_3}{\partial t^2}.$$

Now the differential equation

$$\nabla^2 f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}$$

is the *wave equation* for a wave travelling with speed c. Thus, it follows from Maxwell's equations that in regions of space without charge or current, all components of $\vec{E}$ satisfy the wave equation with speed

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}. \tag{30}$$

A similar argument shows that all components of $\vec{B}$ satisfy the wave equation with the same c.

This led to a profound realization: light (as well as ultraviolet, infrared, radio waves, etc.) is really an electromagnetic wave. Further, the speed of light c can be calculated from the electric and magnetic constants μ_0 and ϵ_0 using equation (30). Thus, one of the most important discoveries of the nineteenth century came from a mathematical reformulation of the laws of electromagnetism.