NOTES ON POWER SERIES

A power series (in x) is a series of the form

$$\sum_{n\geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots, \tag{1}$$

where we adopt the convention $x^0 = 1$. It follows from the ratio test that the power series (1) always converges for $|x| < R$, where R is the radius of convergence:

$$R = 1/L, \text{ where } L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

(here "lim sup" means the least upper bound on the set of limits of subsequences of $\{a_n\}$: unlike the limit, this always exists).

Many power series arise as Maclaurin series of known functions. Recall that *if* the power series (1) represents an infinitely differentiable function $f(x)$ (on some interval around 0), *then* we must have $a_n = f^{(n)}(0)/n!$. Some familiar examples are

$$e^x = \sum_{n\geq 0} \frac{x^n}{n!}, \quad |x| < \infty \tag{2}$$

$$\sin x = \sum_{n\geq 0} \frac{x^{2n+1}}{(2n+1)!}, \quad |x| < \infty \tag{3}$$

$$\cos x = \sum_{n\geq 0} \frac{x^{2n}}{(2n)!}, \quad |x| < \infty \tag{4}$$

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n, \quad |x| < 1 \tag{5}$$

Given this basic stock of power series, we can obtain more by several methods:

1. **Differentiation:** Within its radius of convergence a power series may be differentiated term-by-term. Notice that this just takes equation (2) into itself, and exchanges equations (3) and (4), but applied to equation (5) it gives something new:

$$\frac{1}{(1-x)^2} = \sum_{n\geq 0} n x^{n-1}, \quad |x| < 1. \tag{6}$$

2. **Integration:** We can also integerate a series term-by-term within its radius of convergence. Again, the only time this gives us something new in the examples above is with equation (5):

$$-\ln(1-x) = \sum_{n\geq 0} \frac{x^{n+1}}{n+1} = \sum_{n\geq 1} \frac{x^n}{n}, \quad |x| < 1.$$

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3. **Substitution:** If $g(x)$ is a power series with no constant term and $f(x)$ is represented by the power series (1), then $f(g(x))$ is represented by the power series

$$\sum_{n\geq 0} a_n (g(x))^n.$$

This principle is easiest to use when $g(x)$ consists of just a power of x: for example, from equation (2) above we have

$$e^{x^2} = \sum_{n\geq 0} \frac{x^{2n}}{n!}, \quad |x| < \infty.$$

But other choices for $g(x)$ can be useful. For example, using $g(x) = x + x^2$ in equation (5) gives

$$\begin{aligned}
\frac{1}{1-x-x^2} &= \sum_{n\geq 0} x^n (x+1)^n \\
&= \sum_{n\geq 0} x^n \sum_{i=0} \binom{n}{i} x^i \\
&= \sum_{n\geq 0} \sum_{i\geq 0} \binom{n}{i} x^{i+n} \\
&= \sum_{m\geq 0} \sum_{m-i\geq i} \binom{m-i}{i} x^m \\
&= \sum_{m\geq 0} \sum_{2i\leq m} \binom{m-i}{i} x^m
\end{aligned}$$

for $|x + x^2| < 1$. If you work out the first few coefficients

$$b_m = \sum_{0\leq i\leq m/2} \binom{m-i}{i}, \tag{7}$$

you may find that you've seen them before!

4. **Multiplication:** We can also multiply one power series by another, though this is usually more work than the other methods. The general principle is that if we have power series

$$f(x) = \sum_{n\geq 0} a_n x^n \quad \text{and} \quad g(x) = \sum_{n\geq 0} b_n x^n$$

then

$$f(x)g(x) = \sum_{n\geq 0} c_n x^n, \quad \text{where} \quad c_n = \sum_{i=0}^{n} a_i b_{n-i}.$$

The easiest case is where $g(x)$ is a power of x. For example, from equation (6) above it follows that

$$\frac{x}{(1-x)^2} = \sum_{n\geq 0} nx^n, \quad |x| < 1.$$

Equation (6) itself can be obtained from equation (5) by squaring, that is, taking $f(x) = g(x) = (1-x)^{-1}$. Of course it's easier to get equation (6) by differentiation as we did above.

These methods can be combined. Starting with equation (5), we can substitute $-x^2$ for x to get

$$\frac{1}{1+x^2} = \sum_{n\geq 0} (-1)^n x^{2n}, \quad |x| < 1$$

and then integrate both sides to get

$$\arctan(x) = \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| < 1.$$

Exercise 1. What are the numbers defined by equation (7)?

Exercise 2. Starting with equation (2) above, get a power series for

$$\int_0^x e^{-t^2}\, dt.$$

Exercise 3. What function is represented by the series

$$\sum_{n\geq k} n^{\underline{k}} x^{n-k} = \sum_{n\geq k} n(n-1)\cdots(n-k+1)x^{n-k}?$$

Hint: start with equation (5).