GENERATING FUNCTIONS

Suppose we want to find a formula for the generating function of the Fibonacci numbers,

$$F(t) = \sum_{k\geq 0} F_k t^k = t + t^2 + 2t^3 + 3t^4 + 5t^5 + 8t^6 + \cdots .$$

Then

$$\begin{aligned} F(t) &= t + \sum_{k\geq 2} F_k t^k \\ &= t + \sum_{k\geq 0} F_{k+2} t^{k+2} \\ &= t + \sum_{k\geq 0} (F_k + F_{k+1}) t^{k+2} \\ &= t + \sum_{k\geq 0} F_k t^{k+2} + \sum_{k\geq 0} F_{k+1} t^{k+2} \\ &= t + t^2 F(t) + tF(t) \end{aligned}$$

Hence $(1-t-t^2)F(t) = t$, or

$$F(t) = \frac{t}{1-t-t^2}.$$

Note that the recurrence $F_n = F_{n-1} + F_{n-2}$ can be read off from the denominator: in general, if a sequence S has a rational generating function

$$S(t) = \frac{f(t)}{1 + a_1 t + a_2 t^2 + \cdots + a_p t^p},$$

then $S_n = -a_1 S_{n-1} - a_2 S_{n-2} - \cdots - a_p S_{n-p}$.

Exercise 1. Expanding out $F(t)$ as in the first set of notes, i.e.,

$$\begin{aligned} F(t) &= \frac{t}{1-t(1+t)} \\ &= t \sum_{k\geq 0} t^k (1+t)^k \\ &= t \sum_{k\geq 0} \sum_{j=0}^{k} \binom{k}{j} t^{j+k} \\ &= t \sum_{n\geq 0} \sum_{j} \binom{n-j}{j} t^n, \end{aligned}$$

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prove that

$$F_{n+1} = \sum_i \binom{n-i}{i}. \tag{1}$$

We can also expand $F(t)$ in partial fractions. In doing this, the following computational trick is useful. Let

$$G(t) = F(\frac{1}{t}) = \frac{\frac{1}{t}}{1 - \frac{1}{t} - \frac{1}{t^2}} = \frac{t}{t^2 - t - 1}.$$

Now we can factor $t^2 - t - 1 = (t - \alpha)(t - \beta)$, where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

are the roots of the equation $t^2 - t - 1 = 0$. Then in the partial-fractions decomposition

$$G(t) = \frac{t}{(t-\alpha)(t-\beta)} = \frac{A}{t-\alpha} + \frac{B}{t-\beta}$$

we find that

$$A = \frac{\alpha}{\alpha - \beta} = \frac{\alpha}{\sqrt{5}}, \quad B = \frac{\beta}{\beta - \alpha} = -\frac{\beta}{\sqrt{5}}.$$

Thus

$$\begin{aligned} G(t) &= \frac{\alpha}{\sqrt{5}} \frac{1}{t-\alpha} - \frac{\beta}{\sqrt{5}} \frac{1}{t-\beta} \\ &= \frac{1}{\sqrt{5}} \left[\frac{\alpha}{t-\alpha} - \frac{\beta}{t-\beta} \right] \\ &= \frac{1}{\sqrt{5}} \left[\frac{\alpha/t}{1-\alpha/t} - \frac{\beta/t}{1-\beta/t} \right] \end{aligned}$$

Then using the geometric series we have, at least for small t,

$$\begin{aligned} F(t) &= \frac{1}{\sqrt{5}} \left[\frac{\alpha t}{1-\alpha t} - \frac{\beta t}{1-\beta t} \right] \\ &= \frac{1}{\sqrt{5}} \left[\sum_{k \ge 1} (\alpha t)^k - \sum_{k \ge 1} (\beta t)^k \right] \\ &= \sum_{k \ge 1} \frac{\alpha^k - \beta^k}{\sqrt{5}} t^k. \end{aligned}$$

Since the coefficient of t^n in $F(t)$ is F_n, this means that

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

which is usually known as the Binet formula.

Exercise 2. Recall that the Lucas numbers can be defined by $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$. The corresponding generating function is

$$L(t) = \sum_{k \geq 1} L_k t^k = t + 3t^2 + 4t^3 + 7t^4 + \cdots .$$

a. Show that

$$L(t) = \frac{t + 2t^2}{1 - t - t^2}$$

b. Use partial fractions to write L_n in terms of α and β.

A very useful generating function can be obtained by differentiating the geometric series

$$\frac{1}{1-t} = \sum_{k \geq 0} t^k$$

repeatedly: differentiating n times gives

$$\frac{n!}{(1-t)^{n+1}} = \sum_{k \geq n} k(k-1) \cdots (k-n+1) t^{k-n},$$

or, after multiplying both sides by $t^n/n!$,

$$\frac{t^n}{(1-t)^{n+1}} = \sum_{k \geq n} \binom{k}{n} t^k \tag{2}$$

We shall use this result to find the generating function for the sequence G_n defined by

$$G_n = \sum_k \binom{n-k}{2k}, \ n \geq 0.$$

Our generating function $G(t)$ is

$$\begin{aligned}
\sum_{n\geq 0} G_n t^n &= \sum_{n\geq 0}\sum_{k\geq 0}\binom{n-k}{2k}t^n \\
&= \sum_{k\geq 0}\sum_{n\geq 3k}\binom{n-k}{2k}t^n \\
&= \sum_{k\geq 0}t^k\sum_{n\geq 3k}\binom{n-k}{2k}t^{n-k} \\
&= \sum_{k\geq 0}t^k\sum_{m}\binom{m}{2k}t^m \\
&= \sum_{k\geq 0}t^k\frac{t^{2k}}{(1-t)^{2k+1}} \quad \text{(equation (2))} \\
&= \frac{1}{1-t}\sum_{k\geq 0}\frac{t^{3k}}{(1-t)^{2k}} \\
&= \frac{1}{1-t}\cdot\frac{1}{1-\frac{t^3}{(1-t)^2}} \quad \text{(geometric series)} \\
&= \frac{1-t}{(1-t)^2-t^3} \\
&= \frac{1-t}{1-2t+t^2-t^3}.
\end{aligned}$$

From this we can read off the recurrence $G_n = 2G_{n-1} - G_{n-2} + G_{n-3}$.

Exercise 3. Apply a similar argument to show that the sequence

$$H_n = \sum_k \binom{n-2k}{k}$$

satisfies the recurrence $H_n = H_{n-1} + H_{n-3}$.