COUNTING COMBINATIONS

Suppose you have pennies, nickels, dimes, and quarters. In how many ways can you make change for \$0.43? In a problem like this, we're looking at how many ways a combination can be built up from smaller pieces. Generating functions provide a way to attack this problem.

First, let's consider the problem of making change with just pennies. There is exactly one way to make change for any amount with pennies, so the number P_n of ways of making change for n cents with pennies is $P_n = 1$ for all n, and the corresponding generating function is is

$$P(t) = 1 + t + t^2 + t^3 + \cdots = \frac{1}{1-t}.$$

(The constant term reflects the empty set as a way of making change for zero cents.)

Now suppose you're making change with just nickels. You can only make change for amounts that are a multiple of five, and for such amounts you can do it in only one way: so the number N_n of ways of making change for n cents with nickels is

$$N_n = \begin{cases} 1, & \text{if } n \text{ is a multiple of } 5 \\ 0, & \text{otherwise} \end{cases}$$

and the corresponding generating function is

$$N(t) = 1 + t^5 + t^{10} + t^{15} + \cdots = \frac{1}{1-t^5}.$$

What if you make change with pennies and nickels? Given any amount of n cents, you can make it with n pennies, or $n-5$ pennies and 1 nickel, or $n-10$ pennies and 2 nickels, and so forth. That is, if C_n is the number of ways of making change for n cents using pennies and nickels, then

$$C_n = P_n + P_{n-5}N_5 + P_{n-10}N_{10} + \cdots$$

But this means, if $C(t) = C_0 + C_1 t + C_2 t^2 + \cdots$ is the generating function for making change with pennies and nickels, that

$$C(t) = P(t)N(t) = \frac{1}{(1-t)(1-t^5)}.$$

The point is that multiplication of generating functions reflects the way that combinations are formed. It should be clear what the generating function for making change with pennies, nickels, dimes and quarters is:

$$F(t) = \frac{1}{(1-t)(1-t^5)(1-t^{10})(1-t^{25})}.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

By multiplying this out (most easily using a computer algebra system) one finds that the coefficient of t^{43} is 31, so there are 31 ways to make change for \$0.43.

Let's formalize this a little. We are interested in counting the combinations (let's call them *molecules*) put together from certain basic elements (let's call them *atoms*). Each atom has a *weight*, and the weight of a molecule is the sum of the weights of its atoms. Suppose there are a_n types of atoms of weight n, and let

$$C(t) = 1 + C_1 t + C_2 t^2 + C_3 t^3 + \cdots$$

be the generating function for the number C_n of molecules of weight n. We will show that

$$C(t) = \prod_{n \geq 1} \frac{1}{(1-t^n)^{a_n}}. \tag{1}$$

In fact, we've already introduced the basic ideas of the proof in our change-making example. First, if there is a single type of atom of weight w, the number C_n of molecules is

$$C_n = \begin{cases} 1, & \text{if } n \text{ is a multiple of } w, \\ 0, & \text{otherwise,} \end{cases}$$

and so the corresponding generating function is

$$1 + t^w + t^{2w} + t^{3w} + \cdots = \frac{1}{1-t^w}. \tag{2}$$

The second basic idea is about combining different types of atoms. Let suppose we have two disjoint sets of atoms, say "red" and "green." Let R_n be the number of red molecules of weight n, and

$$R(t) = 1 + R_1 t + R_2 t^2 + \cdots$$

the corresponding generating function. Similarly we have a generating function

$$G(t) = 1 + G_1 t + G_2 t^2 + \cdots$$

that counts green molecules. Now let C_n be the number of molecules of weight n made from both types of atoms. Evidently

$$C_n = R_n + R_{n-1} G_1 + \cdots + R_1 G_{n-1} + G_n$$

which is to say that the generating function $C(t)$ for the C_n is the product of $R(t)$ and $G(t)$.

To prove equation (1), we first observe that it reduces to equation (2) in the case where there is only one type of atom. Now suppose we have a_n types of atom of weight n, and no other atoms. Then applying the idea of the last paragraph we see that the generating function for the number of molecules is

$$\left(\frac{1}{1-t^n}\right)^{a_n} = \frac{1}{(1-t^n)^{a_n}}.$$

Finally, put together atoms of different weights $n = 1, 2, 3, \ldots$, to get equation (1).

Here is an example of how equation (1) can be used to solve a counting problem. Suppose we want to count monomials in three variables x, y, and z by total degree (the total degree of a monomial $x^a y^b z^c$ is $a + b + c$). For example, there are six monomials of total degree two:

$$x^2, \quad y^2, \quad z^2, \quad xy, \quad yz, \quad xz.$$

To apply equation (1), we treat x, y, z, as the atoms, and monomials as molecules. The total degree is the weight. Since x, y, and z all have degree one, the generating function for counting monomials is

$$\frac{1}{(1-t)^3}.$$

But we know that

$$\frac{1}{(1-t)^3} = \sum_{n\geq 0} \binom{n+2}{2} t^n,$$

so the number of monomials of degree n is

$$\binom{n+2}{2} = \frac{(n+1)(n+2)}{2}.$$

Exercise 1. Work out a formula for the number of monomials of total degree n in four variables u, v, w, and x, and use it to compute the number of four-variable monomials of degree 10.

A famous example of formula (1) is the generating function for partitions of integers. A partition of a positive number n is a way of writing it as a sum of positive integers. For example, $n = 5$ has seven partitions:

$$1+1+1+1+1, \quad 2+1+1+1, \quad 3+1+1, \quad 2+2+1, \quad 4+1, \quad 3+2, \quad 5.$$

Here the atoms are the positive integers, the molecules sums of them; the weight of a "molecule" is the sum of its atoms. Since there is one atom of each weight, the generating function for the sequence $p(n)$ that counts partitions of n is

$$P(t) = \sum_{n\geq 0} p(n)t^n = \prod_{n\geq 1} \frac{1}{1-t^n} = \frac{1}{(1-t)(1-t^2)(1-t^3)\cdots}.$$

How can such an infinite product be used for anything? Actually, you never have to multiply out infinitely many factors to find a particular coefficient. Note that the product can be written

$$(1+t+t^2+\cdots)(1+t^2+t^4+\cdots)(1+t^3+t^6+\cdots)\cdots$$

so the nth factor only affects the coefficients of x^n and higher terms. Thus, if we want the coefficient of x^7, it is enough to multiply out the first seven factors. (It's still convenient to use computer algebra: the coefficient of x^7 in

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)}$$

is 15.) We can also study variations on the problem: for example, if $p_{od}(n)$ is the number of partitions of n into odd parts, the corresponding generating function is

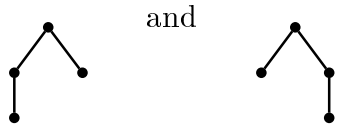
$$P_{od}(t) = \sum_{n\geq 0} p_{od}(n)t^n = \frac{1}{(1-t)(1-t^3)(1-t^5)\cdots}.$$

Exercise 2. Show that the generating function

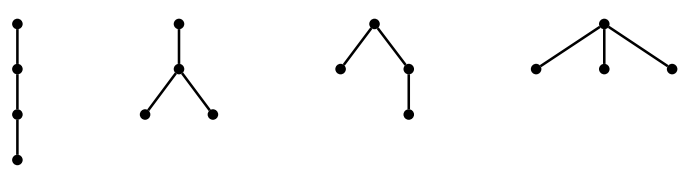
$$P_{ev}(t) = \sum_{n\geq 0} p_{ev}(n)t^n,$$

where $p_{ev}(n)$ is the number of ways to write n as a sum of even numbers, is given by $P_{ev}(t) = P(t^2)$. What is $P_{ev}(t)P_{od}(t)$?

Generating functions can be useful even when there is no simple closed form. For example, we consider the problem of counting rooted trees. Let T_n be the number of rooted trees with n vertices. (We saw earlier that the number of *planar* rooted trees with n vertices is a Catalan number C_{n-1}, but now we're not thinking of the trees as embedded in the plane: we count



as the same rooted tree.) Then $T_4 = 4$, since there are four rooted trees with four vertices:



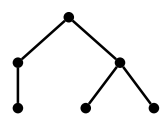
How can we find the T_n? Here is an indirect approach. Let's define a *rooted forest* to be a collection of rooted trees. For example, here's a rooted forest with five vertices:

(3)

Now think of rooted trees as atoms and rooted forests as molecules, with the number vertices giving the weight. If F_n is the number of rooted forests with n vertices, then equation (1) says that

$$\sum_{n\geq 0} F_n t^n = \frac{1}{(1-t)^{T_1}(1-t^2)^{T_2}(1-t^3)^{T_3}\cdots}$$

On the other hand, we have $T_{n+1} = F_n$, because we can always form a rooted tree from a rooted forest by adding a new root and connecting it to the root of each tree in the forest, and conversely we can get a rooted forest from a rooted tree by deleting the root. For example, the rooted forest (3) above corresponds to the rooted tree



Thus, we have

$$T_1 + T_2 t + T_3 t^2 + T_4 t^3 + \cdots = \frac{1}{(1-t)^{T_1}(1-t^2)^{T_2}(1-t^3)^{T_3}\cdots}. \tag{4}$$

This may not seem like progress, since the unknown T_n's appear on both sides. But notice that to find the coefficient of t^n (and thus T_{n+1}), you only need to multiply the first n factors on the right. Starting with $T_1 = 1$ (which really is obvious), we know that

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$$

agrees with the left-hand side of equation (4) through the t term, so $T_2 = 1$. Then

$$\frac{1}{(1-t)(1-t^2)} = 1 + t + 2t^2 + 2t^3 + \cdots$$

is good through the t^2 term, so $T_3 = 2$, and

$$\frac{1}{(1-t)(1-t^2)(1-t^3)^2} = 1 + t + 2t^2 + 4t^3 + \cdots$$

tells us that $T_4 = 4$.

Exercise 3. Carry this process one step further to find T_5.