Algebra of Multiple Zeta Values: Ohno's and Kawashima's Theorems

Michael E. Hoffman

U. S. Naval Academy

Workshop on Multiple Zeta Values, Feynman Diagrams, and Integrals
19 May 2009

Outline

Notation and Recollections

We recall from last time the noncommutative polynomial algebra $\mathfrak{H} = \mathbf{Q}\langle x, y\rangle$ and its two commutative products: shuffle product $\sqcup\!\sqcup$ and harmonic product $*$. We also recall the subalgebras (with respect to either $\sqcup\!\sqcup$ or $*$) $\mathfrak{H}^0 = \mathbf{Q}1 + x\mathfrak{H}y$ and $\mathfrak{H}^1 = \mathbf{Q}1 + \mathfrak{H}y$. The subalgebra $\mathfrak{H}^0$ can be thought of as representing those sequences that have convergent multiple zeta values:

$$\zeta(x^{i_1-1}y\cdots x^{i_k-1}y) = \sum_{n_1>n_2>\dots n_1\ge 1} \frac{1}{n_1^{i_1}n_2^{i_2}\cdots n_k^{i_k}}.$$

As a vector space, $\mathfrak{H}^1$ is the noncommutative polynomial algebra on generators $z_i = x^{i-1}y$. There is an isomorphism from $(\mathfrak{H}^1, *)$ to the algebra QSym of quasi-symmetric functions sending z_i to the ith power-sum $p_i (= M_{(i)})$ and z_1^i to the ith elementary symmetric function $e_i(= M_{(1,1,\dots,1)})$.

Algebraic Structures

In this talk we will look at some classes of identities expressed in terms of algebraic structures on $\mathfrak{H}$ and its subalgebras:

1. The Hopf algebra structure of QSym: essentially this is a coproduct $\Delta : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$ compatible with the algebra structure.
2. The structure of $\mathfrak{H} = \mathbf{Q}\langle x, y\rangle$ as a QSym-module algebra. This means QSym acts on $\mathfrak{H}$ in the "natural" way a Hopf algebra acts on an algebra. This provides a setting for Ohno's Theorem, which in turn implies the Derivation, Sum and Duality Theorems.
3. Linear endomorphisms called "cyclic derivations" on $\mathfrak{H}$, which permit a nice statement of the Cyclic Sum Theorem. Recent work has shown the latter result follows from Kawashima's Theorem, an even more all-encompassing result than Ohno's.

QSym as a Hopf Algebra

$(\mathfrak{H}^1, *) \cong \mathrm{QSym}$ can be given a Hopf algebra structure by giving it the comultiplication

$$\Delta(z_{i_1} z_{i_2} \cdots z_{i_n}) = \sum_{j=0}^{n} z_{i_1} \cdots z_{i_j} \otimes z_{i_{j+1}} \cdots z_{i_n}$$

and the counit ϵ with $\epsilon(u) = 0$ for all elements u of positive degree. This extends the well-known Hopf algebra structure on the algebra Sym of symmetric functions (Geissinger, 1976); note that the power sums $p_i (\leftrightarrow z_i)$ are primitive, i.e.,

$$\Delta(p_i) = p_i \otimes 1 + 1 \otimes p_i.$$

(In a graded connected Hopf algebra like QSym the terms $u \otimes 1$ and $1 \otimes u$ must occur in $\Delta(u)$ for any u of positive degree, so primitive elements have $\Delta(u)$ as simple as possible.)

QSym as a Hopf Algebra cont'd

On the other hand, the elementary symmetric functions $e_i (\leftrightarrow y^i)$ and complete symmetric functions h_i each form a system of divided powers, i.e.,

$$\Delta(e_i) = \sum_{j=0}^{i} e_j \otimes e_{i-j}$$

and similarly for the h_i.

The Hopf algebra $(\mathfrak{H}^1, *, \Delta)$ is commutative but not cocommutative. For example,

$$\Delta(z_1 z_2) = z_1 z_2 \otimes 1 + z_1 \otimes z_2 + 1 \otimes z_1 z_2.$$

Its (graded) dual is the Hopf algebra of noncommutative symmetric functions as defined by Gelfand et. al. (1995).

QSym-Action on $\mathfrak{H}$

Now define $\cdot : \mathfrak{H}^1 \otimes \mathbf{Q}\langle x, y\rangle \to \mathbf{Q}\langle x, y\rangle$ by setting $1 \cdot w = w$ for all words w,

$$z_k \cdot 1 = 0, \quad z_k \cdot x = 0, \quad z_k \cdot y = x^k y$$

for all $k \geq 1$, and

$$u \cdot w_1 w_2 = \sum_u (u' \cdot w_1)(u'' \cdot w_2),$$

where $\Delta(u) = \sum_u u' \otimes u''$; the coassociativity of Δ insures this is well-defined. It turns out that

$$u \cdot w = \text{terms of length } \ell(w) \text{ in } u * w,$$

which implies that $u \cdot (v \cdot w) = (u * v) \cdot w$ for words u, v of $\mathfrak{H}^1$ and $w \in \mathfrak{H}$, so we have the following result.

Ohno's Theorem

Theorem

$(u, v) \to u \cdot v$ defines an action of the Hopf algebra QSym *on* $\mathbf{Q}\langle x, y\rangle$*, and makes* $\mathbf{Q}\langle x, y\rangle$ *a* QSym*-module algebra.*

The action gives us a nice way to state Ohno's 1999 theorem.

Theorem (Ohno, 1999)

For any word $w \in \mathfrak{H}^0$ and nonnegative integer i,

$$\zeta(h_i \cdot w) = \zeta(h_i \cdot \tau(w)).$$

Since $z_1 \cdot u = D(u)$ and $h_1 = z_1$, the cases $i = 0$ and $i = 1$ of Ohno's Theorem are the Duality and Derivation Theorems respectively. Ohno's Theorem also implies the Sum Theorem.

Kaneko's Conjecture

M. Kaneko defined derivations ∂_n of $\mathbf{Q}\langle x, y\rangle$ such that

$$\partial_n(x) = -\partial_n(y) = x(x+y)^{n-1}y,$$

and conjectured that $\partial_n(w) \in \ker \zeta$ for all $w \in \mathfrak{H}^0$ and $n \geq 1$. Since $\partial_1 = \bar{D} - D$, the conjecture holds for $n = 1$, and the case $n = 2$ is easily seen from Ohno's Theorem.

Eventually Kaneko and K. Ihara proved this conjecture by showing it equivalent to Ohno's Theorem. We shall prove this using the action of QSym on $\mathfrak{H}$: extend this to an action of QSym$[[t]]$ on $\mathfrak{H}[[t]]$ in the obvious way. Let

$$H(t) = 1 + h_1 t + h_2 t^2 + \cdots \in \mathrm{QSym}[[t]],$$

and set $\sigma_t(u) = H(t) \cdot u$ for $u \in \mathfrak{H}$.

The Exponentiation Theorem

Then Ohno's Theorem says that $\zeta(\sigma_t(u)) = \zeta(\sigma_t(\tau(u)))$, or $\zeta(\bar{\sigma}_t(u) - \sigma_t(u)) = 0$, where $\bar{\sigma}_t = \tau\sigma_t\tau$. Now σ_t is an automorphism of $\mathfrak{H}^0[[t]]$ (in fact $\sigma_t^{-1}(u) = E(-t)\cdot u$, where $E(t) = 1 + yt + y^2t^2 + \cdots$), so Ohno's Theorem is equivalent to

$$\bar{\sigma}_t\sigma_t^{-1}(u) - u \in \ker\zeta$$

for all $u \in \mathfrak{H}^0[[t]]$. The following result then implies Kaneko's conjecture.

Theorem (Exponentiation)

$$\bar{\sigma}_t\sigma_t^{-1} = \exp\left(\sum_{n=1}^{\infty}\frac{t^n}{n}\partial_n\right). \qquad (1)$$

Exponentiation Theorem cont'd

The Exponentiation Theorem is proved by showing that both sides of equation (1) are the unique automorphism of $\mathfrak{H}[[t]]$ fixing t that

1. takes x to $x(1-ty)^{-1}$, and
2. fixes $x+y$.

That $\bar{\sigma}_t\sigma_t^{-1}$ has the first property follows from the calculation

$$\begin{aligned}\bar{\sigma}_t\sigma_t^{-1}(x) &= \bar{\sigma}_t(E(-t)\cdot x)\\ &= \bar{\sigma}_t(x)\\ &= \tau(H(t)\cdot y)\\ &= \tau(y+txy+t^2x^2y+\cdots)\\ &= x+txy+t^2xy^2+\cdots,\end{aligned}$$

and the second property is similar.

Exponentiation Theorem cont'd

On the other hand, to show that $\exp(\partial_t)$ has these properties, where

$$\partial_t = \sum_{n=1}^{\infty} \frac{t^n}{n} \partial_n,$$

we first note that $\partial_n(x+y) = 0$ for all n implies $\exp(\partial_t)$ fixes $x+y$. To show $\exp(\partial_t)(x) = x(1-ty)^{-1}$, we show that

$$G(s) = \exp(s\partial_t)(x)$$

and

$$x\left(1 - \frac{1-(1-tz)^s}{z} y\right)^{-1}$$

both satisfy the initial-value problem $G'(s) = \partial_t G(s)$, $G(0) = x$; hence $G(1) = \exp(\partial_t)(x) = x(1-ty)^{-1}$.

Exponentiation Theorem cont'd

The derivations ∂_n are related to the derivations D_n defined by $D_n(u) = z_n \cdot u$ as follows. Since

$$\frac{d}{dt} \log H(t) = \frac{H'(t)}{H(t)} = \sum_{n=1}^{\infty} p_n t^{n-1},$$

the map σ_t can also be written

$$\sigma_t = \exp\left(\sum_{n=1}^{\infty} \frac{t^n}{n} D_n \right).$$

Exponentiation Theorem cont'd

Hence the Exponentiation Theorem says that

$$\exp\left(\sum_{n=1}^{\infty}\frac{t^n}{n}\partial_n\right)=\exp\left(\sum_{n=1}^{\infty}\frac{t^n}{n}\bar{D}_n\right)\exp\left(-\sum_{n=1}^{\infty}\frac{t^n}{n}D_n\right),$$

and so the ∂_n can be expressed in terms of the D_n and $\bar{D}_n$ via the Campbell-Baker-Hausdorff formula. For example,

$$\partial_2=\bar{D}_2-D_2-[\bar{D}_1,D_1],$$

and

$$\partial_3=\bar{D}_3-D_3-\frac{3}{4}[\bar{D}_1,D_2]-\frac{3}{4}[\bar{D}_2,D_1]+\frac{1}{4}[[\bar{D}_1,D_1],D_1]-\frac{1}{4}[\bar{D}_1,[\bar{D}_1,D_1]].$$

Cyclic Derivations

Cyclic derivations, originally due to Rota, Sagan and Stein, can be formulated in several ways. What follows is D. Voiculescu's variant. Think of $\mathfrak{H} \otimes \mathfrak{H}$ as a two-sided module over $\mathfrak{H}$ via $a(b \otimes c) = ab \otimes c$ and $(a \otimes b)c = a \otimes bc$. Now define a cyclic derivation to be a composition $\mu\delta$, where $\delta : \mathfrak{H} \to \mathfrak{H} \otimes \mathfrak{H}$ is an ordinary derivation and $\mu(a \otimes b) = ba$.

We will be interested in the case where $\delta = \hat{C}$ is the derivation with $\hat{C}(x) = 0$ and $\hat{C}(y) = y \otimes x$. Letting $C = \mu\hat{C}$, we have, e.g.,

$$\begin{aligned} C(x^3yxy) &= \mu(x^3(y \otimes x)xy + x^3yx(y \otimes x)) \\ &= \mu(x^3y \otimes x^2y + x^3yxy \otimes x) \\ &= x^2yx^3y + x^4yxy. \end{aligned}$$

Cyclic Sum Theorem

The following result (Hoffman and Ohno, 2003) was conjectured by myself and proved by Ohno, using a partial-fractions argument.

Theorem (Cyclic Sum Theorem)

For any word w of $\mathfrak{H}^1$ that is not a power of y,

$$\zeta(C(w) - \tau C \tau(w)) = 0.$$

For example, applying this result to $w = (x^2y)^n$ gives

$$\zeta((x^3y)(x^2y)^{n-1}) = \zeta((x^2y)^n y) + \zeta((xy)(x^2y)^{n-1}(xy)),$$
$$\text{or} \quad \zeta(4, 3, \ldots, 3) = \zeta(3, 3, \ldots, 3, 1) + \zeta(2, 3, \ldots, 3, 2).$$

Cyclic Sum and Sum Theorems

In fact, the Cyclic Sum Theorem implies the Sum Theorem of our last lecture. To see this, let $u = x + ty$. Then the coefficient of t^k in $xu^{n-2}y$ is the sum of all words $w \in \mathfrak{H}^0$ with $|w| = n$ and $\ell(w) = k$. Calculation shows

$$C(u^{n-1}) = (n-1)txu^{n-2}y \quad \text{and}$$
$$\tau C\tau(u^{n-1}) = (n-1)xu^{n-2}y,$$

so the Cyclic Sum Theorem implies that ζ applied to the coefficient of t^{k-1} equals ζ applied to the coefficent to t^k. But this means the results of applying ζ is independent of k, i.e., it is $\zeta(n)$.

Kawashima's Relations

More recently, G. Kawashima has established the following relations. Let ψ be the automorphism of $\mathbf{Q}\langle x, y\rangle$ sending x to $x+y$ and y to $-y$. Then ψ is a linear involution of $\mathfrak{H}$ (but is not a homomorphism for the harmonic product). Let $L_x : \mathfrak{H} \to \mathfrak{H}$ the linear map defined by $L_x(u) = xu$.

Theorem (Kawashima, 2009)

$L_x(\psi(\mathfrak{H}y * \mathfrak{H}y)) \subset \ker\zeta$.

For example,

$$L_x\psi(y * xy) = L_x(-x^2y + y^3) = -x^3y + xy^3$$

so in this case Kawashima's relation is an instance of the Duality Theorem.

Kawashima's Relations Imply Ohno's

Indeed Kawashima's relations imply duality in full generality. As Kawashima proves in his recent paper, even more is true: the Kawashima relations imply the Ohno relations. As we've already seen, the latter imply the Sum Theorem, the Duality Theorem, and (Kaneko-Ihara's generalized) Derivation Theorem. Kawashima leaves open the question of whether his relations imply the Cyclic Sum Theorem, but this has recently been resolved.

Generalized Cyclic Sum Theorem

T. Tanaka and N. Wakabayashi (preprint) have generalized the Cyclic Sum Theorem with the following result, which they show to follow from Kawashima's Theorem. Make $\mathfrak{H}^{\otimes(n+1)}$ a two-sided module over $\mathfrak{H}$ as we did in the case $n = 1$ above, i.e.,

$$a(w_1 \otimes w_2 \otimes \cdots \otimes w_{n+1}) = aw_1 \otimes w_2 \otimes \cdots \otimes w_{n+1}$$

and

$$(w_1 \otimes w_2 \otimes \cdots \otimes w_{n+1})b = w_1 \otimes \cdots \otimes w_n \otimes w_{n+1}b,$$

and let $\mu_n : \mathfrak{H}^{\otimes(n+1)} \to \mathfrak{H}$ be the reverse multiplication map, i.e.,

$$\mu_n(w_1 \otimes w_2 \otimes \cdots \otimes w_{n+1}) = w_{n+1} \cdots w_2 w_1.$$

Generalized Cyclic Sum Theorem cont'd

Now let $\hat{C}_n : \mathfrak{H} \to \mathfrak{H}^{\otimes(n+1)}$ be the derivation with

$$\hat{C}_n(x) = 0 \quad \text{and} \quad \hat{C}_n(y) = y \otimes (x+y)^{\otimes(n-1)} \otimes x,$$

and put $C_n = \mu_n \hat{C}_n$ (so $C_1 = C$, the cyclic derivation defined above). Then the result of Tanaka and Wakabayashi is as follows.

Theorem (Tanaka-Wakabayashi)

For any word w of $\mathfrak{H}^1$ that is not a power of y, $\zeta(C_n(w) - \tau C_n \tau(w)) = 0$ for all $n \geq 1$.

Note the close analogy between this result and the result of Kaneko-Ihara that $\partial_n(w) \in \ker \zeta$ for $w \in \mathfrak{H}^0$.