Outline

Two hook-length formulas

Updown categories

Universal covers of univalent updown categories

A proposed general hook-length formula

Additional examples

Updown Categories as a Framework for Generalized Hook-Length Formulas

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1 Two hook-length formulas

2 Updown categories

3 Universal covers of univalent updown categories

4 A proposed general hook-length formula

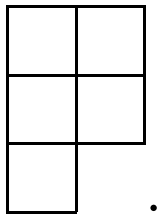
5 Additional examples

Partitions and Young diagrams

An integer partition of n is just a sequence of positive integers that add up to n, regardless of order. For example, 5 has the seven partitions

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1).$$

Partitions of n correspond to conjugacy classes, and thus irreducible representations, of the symmetric group S_n. Each partition has a Young diagram, e.g., $(2,2,1)$ has diagram



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Standard Young tableaux

A standard Young tableau (SYT, plural SYTx) is the Young diagram of a partition of n filled in with the numbers $1, 2, \ldots, n$ so that the numbers increase going right and going downwards. For example, the possible SYTx for $(2, 2, 1)$ are

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}$$

The number K_λ of SYTx for a partition λ of n is of interest since it turns out to be the degree of the irreducible representation of the symmetric group S_n corresponding to λ.

The hook-length formula

The classical hook-length formula of Frame-Robinson-Thrall enumerates the SYTx for a given partition using "hook-lengths". For a partition λ, the hook-length of box in the Young diagram of λ is the number of boxes below it and to the right of it, plus one. Here are the hook-lengths in the Young diagram of $(2, 2, 1)$:

4	2
3	1
1	

For a partition λ, let's write $|\lambda|$ for the sum of the parts of λ, and $Y(\lambda)$ for the Young diagram of λ.

The hook-length formula cont'd

The Frame-Robinson-Thrall hook-length formula says

$$K_\lambda = \frac{|\lambda|!}{\prod_{v \in Y(\lambda)} h(v)},$$

where $h(v)$ is the hook-length of v.
For example, the product of hook-lengths in the diagram of $(2,2,1)$ is $4 \cdot 2 \cdot 3 \cdot 1 \cdot 1 = 24$, so the number of SYTx for $(2,2,1)$ is

$$K_{(2,2,1)} = \frac{5!}{24} = 5.$$

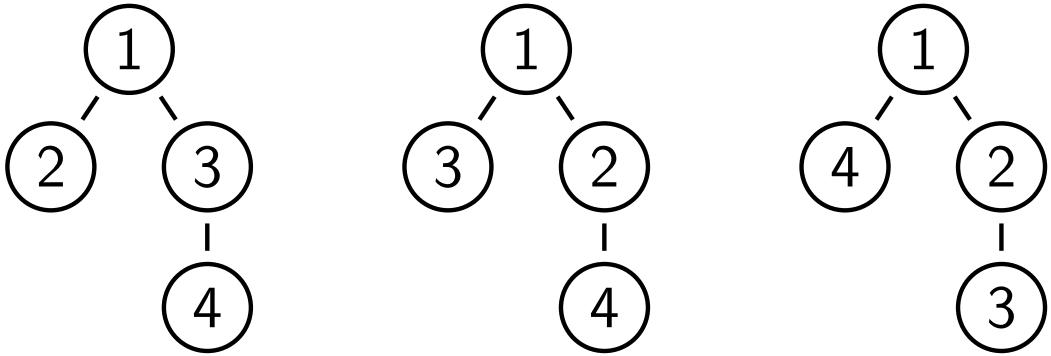
Thus, the five SYTx for $(2,2,1)$ exhibited above are all of them.

Hook-lengths for rooted trees

Let t be a rooted tree with n nodes. A labelled rooted tree (LRT) corresponding to t is an assignment of the numbers $1, 2, \ldots, n$ to the nodes of t so that each node has a label smaller than those of its children. For example, the tree

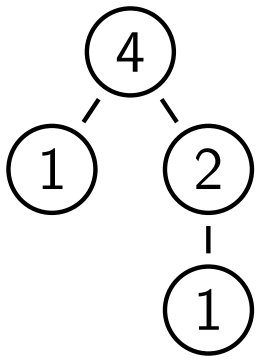
$t =$

has the corresponding LRTs



Hook-lengths for rooted trees cont'd

For node v in a rooted tree, the hook-length $h(v)$ is the number of nodes in the subtree consisting of v and its descendants. For example, the hook-lengths in the tree of the previous slide are



The hook-length formula for rooted trees says that the number of LRTs corresponding to a rooted tree t with n nodes is

$$\frac{n!}{\prod_{v \in t} h(v)}, \quad \text{or in our example,} \quad \frac{4!}{4 \cdot 2 \cdot 1 \cdot 1} = 3.$$

A more general framework

The object of this talk is to present a general framework that encompasses both of the preceding examples, as well as many others. Both partitions and rooted trees are combinatorial objects with a natural rank, and with the property that all objects of rank n can be built up in n steps from one rank-0 object (the empty partition in the first case, the one-node tree in the other).

Combinatorial objects that can be built up in this way constitute what we call an "updown category". Updown categories can be thought of as ranked partially ordered sets with some additional structure. We shall have to make a brief digression into category theory.

Categories in general

A category $\mathcal{C}$ has a class of *objects*. For any two objects x, y of $\mathcal{C}$ there is a set $\mathrm{Hom}(x, y)$ (which may be empty) of *morphisms* from x to y: if $f \in \mathrm{Hom}(x, y)$, then x is the source of f and y is the target of f. If f is a morphism from x to y and g is a morphism from y to z, then there is a composition morphism $g \circ f$ from x to z. One assumes that composition is associative and that the set $\mathrm{Hom}(x, x)$ always contains a morphism id_x such that $id_x \circ f = f$ for any f with target x, and $g \circ \mathrm{id}_x = g$ for any g with source x.

A functor F from category $\mathcal{C}$ to category $\mathcal{D}$ sends each object x of $\mathcal{C}$ to an object $F(x)$ of $\mathcal{D}$, and sends each morphism $f \in \mathrm{Hom}(x, y)$ to a morphism $F(f) \in \mathrm{Hom}(F(x), F(y))$ in a way that respects composition. In particular, $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ and $F(g \circ f) = F(g) \circ F(f)$.

Small categories

There is a category whose objects are sets, and whose morphisms are functions. Similarly, there are categories of groups and group homomorphisms, topological spaces and continuous functions, vector spaces and linear functions, etc. These categories are too "big" for the class of objects to be a set.

A small category is one in which there is a *set* of objects. To take a really small category, suppose there is just one object. Then the set of morphisms from that object to itself is a monoid. Note that a functor from one one-object category to another is a monoid homomorphism. A group can be thought of as a one-object category in which every morphism is invertible.

Updown categories

A partially ordered set can be modeled as a category by making $\mathrm{Hom}(x, y)$ have a single element exactly when $x \leq y$. (Otherwise, $\mathrm{Hom}(x, y)$ is empty.) For example, $\mathbb{N}$ is the category whose object set is $\{0, 1, 2, \dots\}$ and in which $\mathrm{Hom}(i, j)$ has a single morphism if and only if $i \leq j$.
The idea of an updown category is to take this model of a poset and relax the condition that the nonempty sets $\mathrm{Hom}(x, y)$ have only one element. We allow $\mathrm{Hom}(x, x)$ to be a nontrivial group, and if $x < y$ the set $\mathrm{Hom}(x, y)$ may have multiple elements. This allows for automorphism groups and multiplicities respectively. Our precise axioms are as follows.

Axioms for an updown category

$\mathcal{C}$ is a small category, together with a rank functor $|\cdot| : \mathcal{C} \to \mathbb{N}$ so that

1. Each rank $\mathcal{C}_n = \{c \in \operatorname{Ob}\mathcal{C} : |c| = n\}$ is finite.
2. $\mathcal{C}_0$ consists of the single element $\hat{0}$, and $\operatorname{Hom}(\hat{0}, c)$ is nonempty for all $c \in \operatorname{Ob}\mathcal{C}$.
3. The set $\operatorname{Hom}(p, p')$ is always finite, and is empty unless $|p| < |p'|$ or $p = p'$; in the latter case, $\operatorname{Hom}(p, p')$ is a group $\operatorname{Aut}(p)$.
4. Any morphism $p \to p'$, $|p'| = |p| + k$, is a composition $p = p_0 \to p_1 \to \cdots \to p_k = p'$ of morphisms between adjacent ranks.
5. If $|p'| = |p| + 1$, the actions of $\operatorname{Aut}(p)$ and $\operatorname{Aut}(p')$ on $\operatorname{Hom}(p, p')$ are free.

Defining multiplicities

For an updown category $\mathcal{C}$ and objects c, c' of $\mathcal{C}$ with $|c'| = |c| + 1$, we define

$$u(c; c') = |\operatorname{Hom}(c, c')/\operatorname{Aut}(c')| = \frac{|\operatorname{Hom}(c, c')|}{|\operatorname{Aut}(c')|}$$

and

$$d(c; c') = |\operatorname{Hom}(c, c')/\operatorname{Aut}(c)| = \frac{|\operatorname{Hom}(c, c')|}{|\operatorname{Aut}(c)|}.$$

It follows immediately that

$$u(c; c')|\operatorname{Aut}(c')| = d(c; c')|\operatorname{Aut}(c)|. \tag{1}$$

If all the automorphism groups are trivial, then

$$u(c; c') = d(c; c') = |\operatorname{Hom}(c, c')|$$

and we call $\mathcal{C}$ univalent.

Up and down operators

Now let $\mathcal{C}$ be an updown category, k a field of characteristic 0, and $k\,\mathrm{Ob}\,\mathcal{C}$ the graded vector space with elements of $\mathrm{Ob}\,\mathcal{C}$ as basis. We define operators U and D on $k\,\mathrm{Ob}\,\mathcal{C}$ as follows:

$$Uc = \sum_{|c'|=|c|+1} u(c;c')c'$$

for $c \in \mathrm{Ob}\,\mathcal{C}$, and

$$Dc = \sum_{|c'|=|c|-1} d(c';c)c'$$

for $c \in \mathrm{Ob}\,\mathcal{C}$, $c \neq \hat{0}$; we set $D\hat{0} = 0$.

An Inner Product

Now define an inner product $\langle \cdot, \cdot \rangle$ on $k \operatorname{Ob} \mathcal{C}$ by setting

$$\langle c, d \rangle = \begin{cases} |\operatorname{Aut}(c)|, & \text{if } d = c, \\ 0, & \text{otherwise.} \end{cases}$$

Then if $|y| = |x| + 1$,

$$\langle U(x), y \rangle = \langle \sum u(x; x')x', y \rangle = u(x; y) |\operatorname{Aut} y|$$

while

$$\langle x, D(y) \rangle = \langle x, \sum d(y'; y)y' \rangle = d(x; y) |\operatorname{Aut} x|,$$

so U and D are adjoint by equation (1).

Extending the multiplicities

We can now extend the multiplicities $u(c;c')$ and $d(c;c')$ to all pairs of elements $c, c' \in \operatorname{Ob} \mathcal{C}$: suppose $|c'| = |c| + k$. If $k < 0$ we put $u(c;c') = d(c;c') = 0$, while for $k \geq 0$

$$u(c;c') = \frac{\langle U^k(c), c' \rangle}{|\operatorname{Aut}(c')|} = \frac{\langle c, D^k(c') \rangle}{|\operatorname{Aut}(c')|}$$

and

$$d(c;c') = \frac{\langle U^k(c), c' \rangle}{|\operatorname{Aut}(c)|} = \frac{\langle c, D^k(c') \rangle}{|\operatorname{Aut}(c)|}.$$

Then equation (1) extends to arbitrary pairs of objects. In particular,

$$\frac{d(\hat{0};c)}{u(\hat{0};c)} = \frac{|\operatorname{Aut}(c)|}{|\operatorname{Aut}\hat{0}|} \tag{2}$$

for any $c \in \operatorname{Ob} \mathcal{C}$.

Multiplicities cont'd

It is also easy to prove that

$$u(c;c') = \sum_{|c''|=k} u(c;c'')u(c'';c')$$

for any $|c| \leq k \leq |c'|$, and similarly for d.
An updown category is called unital if $u(c;c') = d(c;c')$ is either 0 or 1 for any c, c' with $|c'| = |c| + 1$. Unital implies univalent but not conversely.

Covering spaces

A morphism of updown categories is a little complicated to describe in general, but if $\mathcal{C}$ and $\mathcal{D}$ are univalent it is a functor $F : \mathcal{C} \to \mathcal{D}$ such that $|F(c)| = |c|$ for all $c \in \operatorname{Ob} \mathcal{C}$ and the induced map

$$\coprod_{\{c' | F(c') = F(c)\}} \operatorname{Hom}(b, c') \to \operatorname{Hom}(F(b), F(c)) \tag{3}$$

is injective for all $b, c \in \operatorname{Ob} \mathcal{C}$ with $|c| = |b| + 1$.
We say that a morphism $F : \mathcal{C} \to \mathcal{D}$ of univalent updown categories is a covering map if F is surjective on object sets and the induced function (3) is a bijection for all $b, c \in \operatorname{Ob} \mathcal{C}$ with $|c| = |b| + 1$.

Universal covers

A covering map $F : \tilde{\mathcal{C}} \to \mathcal{C}$ of univalent updown categories is universal if for any other covering map $G : \mathcal{D} \to \mathcal{C}$ there is a covering map $H : \tilde{\mathcal{C}} \to \mathcal{D}$ with $F = GH$. Less formally, $\tilde{\mathcal{C}}$ is the "unfolding" of $\mathcal{C}$ that makes its Hasse diagram a tree.

There is a canonical way to construct the universal cover $\tilde{\mathcal{C}}$ of a univalent updown category $\mathcal{C}$: the rank-n objects of $\tilde{\mathcal{C}}$ are strings

$$\hat{0} \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} c_n \tag{4}$$

with $c_i \in \mathcal{C}_i$ (i.e., paths in the Hasse diagram of $\mathcal{C}$). Morphisms in $\tilde{\mathcal{C}}$ are inclusions of strings. The covering map $P : \tilde{\mathcal{C}} \to \mathcal{C}$ sends the string (4) to c_n, and P sends the inclusion

$$(f_1, \ldots, f_j) \subset (f_1, \ldots, f_n) \quad \text{to} \quad f_n f_{n-1} \cdots f_{j+1} \in \mathrm{Hom}(c_j, c_n).$$

Universal covers cont'd

Given any updown category, we can get univalent updown categories $\mathcal{C}^{\uparrow}$ and $\mathcal{C}^{\downarrow}$ by only using the multiplicities $u(\cdot;\cdot)$ and $d(\cdot;\cdot)$ respectively. Since for $c_n \in \mathcal{C}_n$ we have

$$d(\hat{0}; c_n) = \sum_{(\hat{0}, c_1, \ldots, c_n), c_i \in \mathcal{C}_i} d(\hat{0}; c_1) d(c_1; c_2) \cdots d(c_{n-1}; c_n),$$

we see that $d(\hat{0}; c_n)$ counts paths in the Hasse diagram of $\mathcal{C}^{\downarrow}$. Then we have the following result.

Proposition

Let $\mathcal{C}$ be an updown category. If $P : \tilde{\mathcal{C}}^{\downarrow} \to \mathcal{C}^{\downarrow}$ is the universal cover, the number of elements in $\operatorname{Ob} \tilde{\mathcal{C}}^{\downarrow}$ that P sends to $c \in \operatorname{Ob} \mathcal{C}^{\downarrow} = \operatorname{Ob} \mathcal{C}$ is $d(\hat{0}; c)$.

A general hook-length formula

We propose a general form of the hook-length formula for updown categories with some additional structure on the objects. The conclusion we want is

$$d(\hat{0}; c) = \frac{|c|!}{\prod_{v \in c} h(v)} \tag{5}$$

for some appropriate definition of hook-lengths in c. We show how this fits the hook-length formulas for both SYTx and LRTs.

Partitions as an updown category

Let $\mathcal{P}$ be the updown category such that $\mathcal{P}_n$ is the set of partitions of n, i.e., the rank is

$$|(\lambda_1, \ldots, \lambda_l)| = \lambda_1 + \cdots + \lambda_l.$$

If $|\mu| = |\lambda| + 1$, then $\mu \geq \lambda$ if (a) μ comes from λ by adding a new part of size 1, or (b) μ comes from λ by increasing a part of λ in size. In both cases we put

$$d(\lambda, \mu) = u(\lambda, \mu) = 1,$$

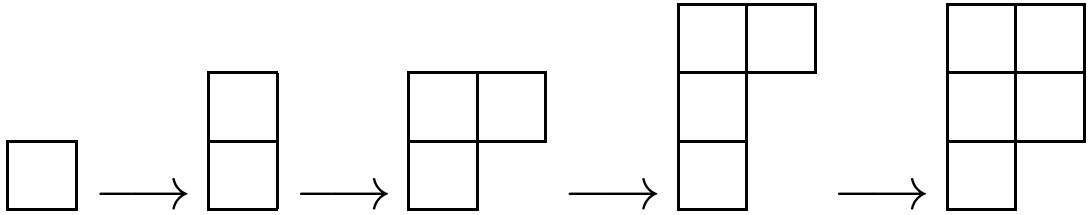
so $\mathcal{P}$ is a unital updown category.

SYTx as covering space elements

A SYT like

1	3
2	5
4	

can be thought of as representing the string



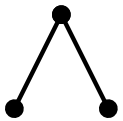
and therefore an element of $\tilde{\mathcal{P}}$. Hence $d(\hat{0}, \lambda)$ is the number of SYTx corresponding to λ, and the Frame-Robinson-Thrall hook-length formula is an instance of (5).

Rooted trees as an updown category

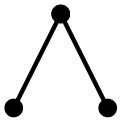
Let $\mathcal{T}$ be the set of rooted trees, with rank function

$$|t| = \text{the number of non-root nodes of } t.$$

If $|t'| = |t| + 1$, then t' covers t if and only if a new edge and terminal node can be attached to t to give t'. Let $u(t; t')$ be the number of nodes of t to which a new edge and terminal node can be added to get t', and $d(t; t')$ the number of terminal edges of t' that, when deleted, leave t. For example,

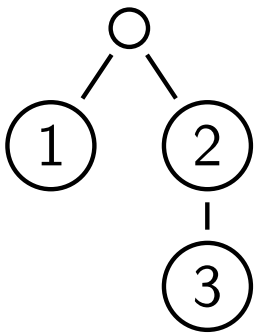
$u($;  $) = 1$

while

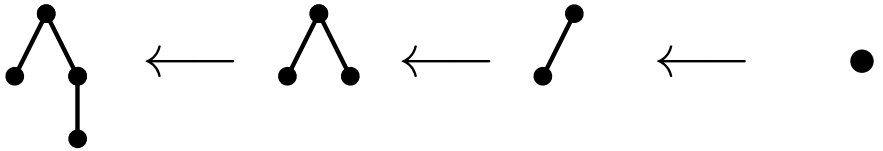
$d($;  $) = 2.$

Rooted trees and LRTs

Because the rank of a rooted tree is the number of non-root nodes, we have to adjust our notions of LRT and hook-lengths slightly. For a LRT, we label the non-root nodes of $t \in \mathcal{T}_n$ with the numbers $1, 2, \ldots, n$ so that the label of each node is smaller than those of its children. Then each LRT describes a path in $\mathcal{T}^{\downarrow}$: we remove the largest label, one at a time, to get back to the root. For example,



encodes

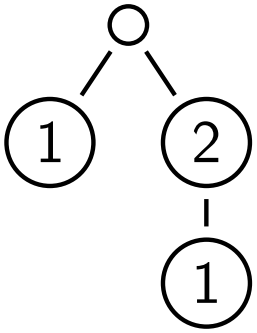


Rooted trees and LRTs cont'd

We only assign hook-lengths to non-root nodes, so hook-lengths of

$t =$

are



Since $|t| = 3$, we still have

$$\frac{|t|!}{\prod_{v \in t} h(v)} = \frac{3!}{1 \cdot 2 \cdot 1} = 3.$$

Thus, the hook-length formula for rooted trees is also an instance of (5).

Additional examples: factorial updown categories

An updown poset $\mathcal{C}$ is called factorial if

$$\sum_{|c'|=|c|-1} d(c';c) = |c|$$

for every $c \in \operatorname{Ob}\mathcal{C}$. It follows that $d(\hat{0};c) = |c|!$ for every such c, so equation (5) holds if all hook-lengths are taken to be 1. Interesting examples of factorial updown posets include

- the subsets of $\{1, 2, \ldots, n\}$;
- monomials in any fixed number of variables;
- finite graphs;
- necklaces with beads of any fixed number of colors.

Additional examples: $\mathcal{K}$, the "other" partition UC

Let $\mathcal{K}$ be the updown category such that $\mathcal{K}_n = \mathcal{P}_n$ is the set of partitions of n. For any partition λ, let $m_k(\lambda)$ be the number of parts of size k in λ. If $|\mu| = |\lambda| + 1$, then μ covers λ if (a) μ comes from λ by adding a new part of size 1, or (b) μ comes from λ by increasing a part of λ in size. In case (a),

$$u(\lambda;\mu) = 1 \quad \text{and} \quad d(\lambda;\mu) = m_1(\mu).$$

In case (b),

$$u(\lambda;\mu) = m_k(\lambda) \quad \text{and} \quad d(\lambda;\mu) = m_{k+1}(\mu),$$

where μ is gotten by increasing a size-k part of λ to size $k+1$. The automorphism group of $\lambda \in \operatorname{Ob}\mathcal{K}$ is the group exchanging parts of λ of equal size: it has order

$$m_1(\lambda)!\,m_2(\lambda)!\cdots.$$

Elements of the covering space $\mathcal{K}^{\downarrow}$

An element of the covering space $\mathcal{K}^{\downarrow}$ mapping to $\lambda \in \mathcal{K}_n$ can be identified with the Young diagram of λ with the boxes filled in by $1, 2, \ldots, n$ so that the labels increase along the rows. There are twelve such elements corresponding to $(2, 1, 1)$:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

(Remark: only the first three of these are SYTx.)

Hook-length formula in $\mathcal{K}$

The definition of hook-lengths in $\mathcal{K}$ is similar to that in $\mathcal{P}$, but now the hook-length of a box in the Young diagram of λ is just one plus the number of boxes to its right. For $\lambda = (2, 1, 1)$ the hook-lengths are

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

and so the hook-length formula gives

$$\frac{4!}{2} = 12.$$

Yet more examples

Other examples of updown categories that admit hook-lengths making equation (5) true are the updown categories of

- integer compositions (ordered partitions);
- planar rooted trees;
- planar binary rooted trees.