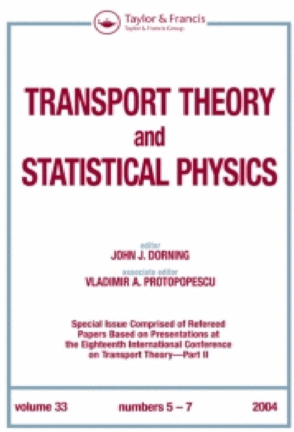

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GLOBAL EXISTENCE AND INCREASED SPATIAL DECAY FOR THE RADIAL VLASOV-POISSON SYSTEM WITH STEADY SPATIAL ASYMPTOTICS

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A collisionless plasma is modeled by the Vlasov-Poisson system in three space dimensions. A fixed background of positive charge, which is independent of time and space, is assumed. The situation in which mobile negative ions balance the positive charge as $|x| \to \infty$ is considered. Hence the total positive charge, total negative charge, and total energy are infinite. Smooth solutions with appropriate asymptotic behavior for large $|x|$, which were previously shown to exist locally in time, are continued globally for spherically symmetric data. This is done by showing that the charge density decays at least as fast as $|x|^{-4}$. Finally, an increased decay rate of $|x|^{-6}$ is shown in the general case without the assumption of spherical symmetry.

1. Introduction

Let $F : \mathbb{R}^3 \to [0, \infty)$, $f_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$, and $A : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ be given. We seek a solution, $f : [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ satisfying

$$\left.\begin{array}{c} \partial_t f + v \cdot \nabla_x f - (E + A) \cdot \nabla_v f = 0, \\ \rho(t, x) = \int (F(v) - f(t, x, v))dv, \\ E(t, x) = \int \rho(t, y) \dfrac{x - y}{|x - y|^3} dy, \\ f(0, x, v) = f_0(x, v). \end{array}\right\} \tag{1}$$

Here F describes a number density of positive ions that form a fixed background, and f denotes the density of mobile negative

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ions in phase space. Notice that if $f_0(x, v) = F(v)$ and $A = 0$, then $f(t, x, v) = F(v)$ is a steady solution. Thus, we seek solutions for which $f(t, x, v) \to F(v)$ as $|x| \to \infty$. Precise conditions that ensure local existence were given by Schaeffer (2003b). It is important to notice that (1) is a representative problem, and that problems concerning multiple species of ions can be treated in a similar manner.

The paper will be divided into two main results. The first will be devoted to showing the global existence of a smooth solution to (1) in the case of spherically symmetric data. This serves to continue the local existence result of Schaeffer (2003b). The second, then, is devoted to achieving an increased rate of spatial decay of the charge density without the spherical symmetry assumption. We will assume throughout that F has compact velocity support, so that decay in v of the background is not an issue. Thus, the main difficulty in showing global existence arises in showing that ρ decays rapidly enough in $|x|$. To see this, consider the following heuristic argument, discussed by Schaeffer (2003b). Let $r = |x|$. Then, we typically expect the electric field, E, to decay like r^{-2} for large r. If we let $g = F - f$, then

$$\partial_t g + v \cdot \nabla_x g - (E + A) \cdot \nabla_v g = -(E + A) \cdot \nabla_v F. \qquad (2)$$

Viewing $(E + A)\cdot\nabla_v F$ as a source term for g, we can only conclude r^{-2} decay for both g and ρ. But, if ρ really decayed like r^{-2}, the integral for E could certainly fail to decay like r^{-2}. In fact, we cannot show that g decays faster than r^{-2}, but due to cancellation in the v integral, ρ must decay faster.

The Vlasov-Poisson system has been studied extensively in the case when $F(v)=0$ and solutions decay to 0 as $|x| \to \infty$. Smooth solutions were shown to exist globally in time by Pfaffelmoser (1992) and independently by Lions and Pertham (1991). Important results prior to global existence were given by Batt (1977), Horst (1981, 1982), Kurth (1952), and Glassey and Strauss (1986). Also, the method used by Pfaffelmoser (1992) has been refined by Horst (1993) and Schaeffer (1991). The global existence of the Vlasov-Poisson system in two dimensions was established by Okabe and Ukai (1978) and Wollman (1986). A complete discussion of the literature concerning Vlasov-Poisson may be found in Glassey (1996). We also mention Batt and Rein

(1991) since the problem treated in said paper is periodic in space, and thus the solution does not decay for large $|x|$. The works cited above make extensive use of the laws of conservation of charge and energy. However, in the problem considered here, and those of Caglioti, et al. (2001), Jabin (2001), and Schaeffer (2003a, b), the charge and energy are infinite, and it is less clear how to use the conservation laws. Therefore, the use of conservation laws will be an important issue, and we will utilize a lemma (stated here as Lemma 2) from Schaeffer (2003a) to deal with it properly.

2. Preliminaries

We will use notation that follows Schaeffer (2003b). For $q > 7 + \sqrt{33}$, let

$$p = 4 - \frac{8}{q}$$

and denote

$$R(x) = R(|x|) = (1 + |x|^2)^{1/2}.$$

We will use the norms

$$\begin{aligned} \|g\|_\infty &= \sup_{z \in \mathbb{R}^3} |g(z)| \\ \|\rho\|_p &= \|\rho R^p(x)\|_\infty, \\ \|g\|_q &= \|g(1 + |x|^2 + |v|^q)\|_{L^\infty(\mathbb{R}^6)}, \end{aligned}$$

and

$$|||g||| = \|g\|_q + \|\nabla g\|_q + \left\| \int g dv \right\|_p,$$

but never use L^p or L^q for finite p and q. We will write, for example, $\|g(t)\|_q$ for the $\|\cdot\|_q$ norm of $(x, v) \mapsto g(t, x, v)$. Notice that we may take q arbitrarily large since we take the initial data to be of compact support.

Following Schaeffer (2003a, b), we assume the following conditions hold for some $C > 0$ and all $t \geq 0$, $x \in \mathbb{R}^3$, and $v \in \mathbb{R}^3$, unless otherwise stated:

(I) $F(v) = F_R(|v|)$ is nonnegative and C^2, with

$$F_R''(0) < 0. \tag{3}$$

In addition, there is $W \in (0, \infty)$ such that

$$\left.\begin{array}{ll} F_R'(u) < 0 & \text{for } u \in (0, W) \\ F_R(u) = 0 & \text{for } u \geq W. \end{array}\right\} \tag{4}$$

(II) f_0 is C^1 with compact v-support, nonnegative, and satisfies the condition of spherical symmetry,

$$f_0(x, v) = f_R(|x|, |v|, x \cdot v), \tag{5}$$

which is equivalent to

$$f_0(x, v) = f_0(Ux, Uv)$$

for every rotation U.

Also, there is $N > 0$ such that for $|x| > N$, we have

$$f_0(x, v) = F(v).$$

(III) A is C^1 and

$$|A(t, x)| + |\partial_x A(t, x)| \leq CR^{-2}(x)$$
$$|\nabla_x \cdot A(t, x)| \leq CR^{-4}(x).$$

Furthermore, A satisfies the condition of spherical symmetry,

$$A(t, x) = \alpha(t, |x|)\frac{x}{|x|^2}, \tag{6}$$

which is equivalent to

$$A(t, x) = U^T A(t, Ux)$$

for every rotation U.

Finally, we assume that there is a continuous function $a : [0, T] \to \mathbb{R}$ such that for $|x| > N$

$$\left|\alpha(t, |x|) - \frac{a(t)}{|x|}\right| \leq R^{2-p}(x).$$

It should be noted that the assumptions (5) and (6) imply the spherical symmetry of $f(t, x, v)$, $g(t, x, v)$, and $E(t, x)$ for all $t \in [0, \infty)$, $x \in \mathbb{R}^3$, and $v \in \mathbb{R}^3$. Thus, where necessary we will write $g(t, x, v) = g(t, |x|, |v|, x \cdot v)$ when using the spherical symmetry of g. Furthermore, the spherical symmetry of E and f will be instrumental in the first results of the paper, while the last theorem is dedicated to eliminating the symmetry assumptions to conclude similar decay rates. We wish to prove the following.

Theorem 1. *Assuming conditions (I), (II), and (III) hold, there exists* $f \in \mathscr{C}^1([0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3)$ *that satisfies* (1) *with* $|||(F - f)(t)|||$ *bounded on* $t \in [0, T]$, *for every* $T > 0$. *Moreover, f is unique.*

In Schaeffer (2003b), both local existence and a criteria for continuation of a bounded, unique solution of (1) are shown. Thus, to prove Theorem 1 we will need to establish the continuation criteria for all $T > 0$. Specifically, Theorems 2 and 3 of Schaeffer (2003b), when combined, state the following.

Theorem 2. *Assume* $q > 7 + \sqrt{33}$ *and conditions (I), (II), and (III) hold, without* (5) *and* (6). *Let f be a* $\mathscr{C}^1$ *solution of* (1) *on* $[0, T] \times \mathbb{R}^3 \times \mathbb{R}^3$ *with* $T > 0$. *If* $\| \int (F - f)(t)\, dv \|_p$ *is bounded on* $[0, T]$, *then we may uniquely extend the solution to* $[0, T + \delta]$ *for some* $\delta > 0$ *with* $|||(F - f)(t)|||$ *bounded on* $[0, T + \delta]$.

Therefore, to prove Theorem 1, we will find a solution to (1) on $[0, T]$ for some $T > 0$ using the local existence theorem (again, see Schaeffer (2003b)), which has been uniquely extended using Theorem 2. Since this may be done as long as the p-norm stays bounded, we will only need to prove the following lemma to establish Theorem 1.

Lemma 1. *Assume* $q > 7 + \sqrt{33}$ *and conditions (I), (II), and (III) hold. Let f be a* $\mathscr{C}^1$ *solution of* (1) *on* $[0, T] \times \mathbb{R}^3 \times \mathbb{R}^3$. *Then,*

$$\left\| \int (F - f)(t) dv \right\|_p \leq C$$

for all $t \in [0, T]$, *where C is determined by F, A,* f_0, *and T.*

Although it is not explicitly stated in Schaeffer (2003b), the proof presented there shows that δ in Theorem 2 is bounded away from 0 as long as $\| \int (F - f)(t)\, dv \|_p$ is bounded. Using this observation, Theorem 1 follows from Lemma 1.

Once Lemma 1 has been established, we will show increased decay of the charge density, ρ, under slightly modified assumptions. Instead of (III), we will assume the following conditions for some $C > 0$ and all $t \geq 0$, $x \in \mathbb{R}^3$, and $v \in \mathbb{R}^3$:

(IV) A is C^1 with

$$|A(t,x)| \leq CR^{-2}(x), \tag{7}$$

$$|\partial_x A(t,x)| \leq CR^{-3}(x), \tag{8}$$

and

$$|\nabla_x \cdot A(t,x)| = 0. \tag{9}$$

Notice that condition (IV) does not involve spherical symmetry of A, and thus the result that follows requires no such assumption.

Theorem 3. *Assume conditions (I) and (II) hold, without* (5), *and condition (IV) holds. Let* $T > 0$ *and* f *be the* $\mathscr{C}^1$ solution to (1) *with*

$$|||(F - f)(t)||| < \infty$$

for every $t \in [0, T)$. *Then, we have*

$$\|\rho(t)\|_6 \leq C_{p,t}$$

for any $t \in [0, T)$, *where* $C_{p,\,t}$ *depends upon*

$$\sup_{\tau \in [0,t]} \|\rho(\tau)\|_p.$$

In Section 3 we will prove Lemma 1, and thus Theorem 1. Then, Section 4 will contain the proof of Theorem 3. We will denote by C a generic constant that changes from line to line and may depend upon f_0, A, F, or T, but not on t, x, or v. When it is necessary to refer to a generic constant that depends upon other variables, we will use variable subscripts to distinguish them. For example, we will use $C_{p,t}$ quite frequently in Section 4, and it will always denote dependence upon $\|\rho(t)\|_p$ and t. When it is necessary to refer to a specific constant, we will use numeric superscripts to distinguish them. For example, $C^{(1)}$ will always refer to the same constant.

3. Global Existance in the Radial Case

Define the characteristics $X(s, t, x, v)$ and $V(s, t, x, v)$ by

$$\left.\begin{aligned} \frac{\partial X}{\partial s}(s,t,x,v) &= V(s,t,x,v) \\ \frac{\partial V}{\partial s}(s,t,x,v) &= -(E(s,X(s,t,x,v)) + A(s,X(s,t,x,v))) \\ X(t,t,x,v) &= x \\ V(t,t,x,v) &= v. \end{aligned}\right\} \tag{10}$$

Then, we have

$$\begin{aligned} \frac{d}{ds} f\,(s, X(s,t,x,v), V(s,t,x,v)) &= \partial_t f + V \cdot \nabla_x f \\ &\quad - (E + A) \cdot \nabla_v f = 0. \end{aligned}$$

Therefore, f is constant along characteristics, and

$$\begin{aligned} f\,(t,x,v) &= f\,(0, X(0,t,x,v), V(0,t,x,v)) \\ &= f_0(X(0,t,x,v), V(0,t,x,v)). \end{aligned} \tag{11}$$

Thus, we find by (II) that f is nonnegative and $\sup_{x,v}|f| = \|f_0\|_\infty < \infty$. Unless necessary, we will omit writing the dependence of $X(s)$ and $V(s)$ on t, x, and v for the remainder of the paper.

In order to bound the electric field, and thus the velocity support, we must use Lemma 3 and Theorem 4 from Schaeffer (2003a). In particular, we state the following lemma without proof.

Lemma 2. *Assuming conditions (I) and (III) hold, without* (6), *there exists* $k : [0, T] \times \mathbb{R}^3 \to [0, \infty)$ *such that*

$$|\rho(t,x)| \leq C(k(t,x)^{3/5} + k(t,x)^{1/2})$$

and

$$\int k(t,x)dx \leq C.$$

Furthermore, notice that, due to the radial symmetry, we may write

$$E(t,x) = \frac{m(t,|x|)}{|x|^2}\frac{x}{|x|} \tag{12}$$

where the enclosed charge is given by

$$m(t,r) := \int_{|y|<r} \rho(t,y)dy.$$

Lemma 2 will be exactly what is needed to bound the electric field, E. We have for any $x \in \mathbb{R}$

$$\begin{aligned}\int_{|y|\leq|x|} k(t,y)^{1/2}dy &\leq \left(\int_{|y|\leq|x|} k(t,y)dy\right)^{1/2}\left(\int_{|y|\leq|x|} dy\right)^{1/2}\\ &\leq \left(\int_{|y|\leq|x|} k(t,y)dy\right)^{1/2}(C|x|^3)^{1/2}\\ &\leq C|x|^{3/2}\end{aligned}$$

and

$$\begin{aligned}\int_{|y|\leq|x|} k(t,y)^{3/5}dy &\leq \left(\int_{|y|\leq|x|} k(t,y)dy\right)^{3/5}(C|x|^3)^{2/5}\\ &\leq C|x|^{6/5}.\end{aligned}$$

Now, we use the bounds on k to estimate the enclosed charge:

$$\begin{aligned}|m(t,|x|)| &\leq \int_{|y|\leq|x|} |\rho(t,y)|dy \\ &\leq C\int_{|y|\leq|x|} (k(t,y)^{3/5} + k(t,y)^{1/2})dy \\ &\leq C(|x|^{3/2} + |x|^{6/5}).\end{aligned}$$

Thus,

$$\begin{aligned}|E(t,x)| &\leq C|m(t,|x|)||x|^{-2} \\ &\leq C|x|^{-2}(|x|^{6/5} + |x|^{3/2}) \\ &\leq C(|x|^{-4/5} + |x|^{-1/2}).\end{aligned}$$

So,

$$|E(t,x)| \leq C\mathcal{G}(|x|),$$

and, combining this with (III),

$$|E(t,x) + A(t,x)| \leq C\mathcal{G}(|x|)$$

where

$$\mathcal{G}(r) := \begin{cases} r^{-4/5}, & r \leq 1 \\ r^{-1/2}, & r \geq 1. \end{cases}$$

Next, we use methods from Horst (1982) to estimate the velocity support. Assume

$$f(t,x,v) = f(0, X(0,t,x,v), V(0,t,x,v)) \neq 0$$

so that, using the compact support in v of f_0, we have

$$|\dot{X}(0)| = |V(0)| \leq C.$$

For any $\varepsilon > 0$, define $a := 2 + \varepsilon$, $b := a/a - 1$, and

$$B := \left(2\int_1^\infty \mathcal{G}(|x|)^a dx\right)^{1/a}.$$

Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, and assume $\dot{X}_i \geq 0$ on $[t_1, t_2]$. We claim

$$\int_{X_i(t_1)}^{X_i(t_2)} \mathcal{G}(|x|)dx \leq B(X_i(t_2) - X_i(t_1))^{1/b} + 10. \tag{13}$$

To show (13), let

$$\mathcal{G}_1(x) := \begin{cases} |x|^{-4/5} & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

and

$$\mathcal{G}_2(x) := \begin{cases} 0 & \text{if } |x| \leq 1 \\ |x|^{-1/2} & \text{if } |x| > 1. \end{cases}$$

Then,

$$\begin{aligned}
\int_{X_i(t_1)}^{X_i(t_2)} \mathcal{G}(|x|)dx &= \int_{X_i(t_1)}^{X_i(t_2)} \mathcal{G}_1(x)dx + \int_{X_i(t_1)}^{X_i(t_2)} \mathcal{G}_2(x)dx \\
&\leq \int_{-1}^{1} \mathcal{G}_1(x)dx + \left(\int_{X_i(t_1)}^{X_i(t_2)} (\mathcal{G}_2(x))^a dx\right)^{1/a} \\
&\quad (X_i(t_1) - X_i(t_2))^{1/b} \\
&\leq 10 + \left(2\int_1^\infty (\mathcal{G}_2(x)\right)^a dx)^{1/a} (X_i(t_1) - X_i(t_2))^{1/b} \\
&\leq 10 + B(X_i(t_1) - X_i(t_2))^{1/b}.
\end{aligned}$$

This establishes (13). Now, using this result and following Horst (1982), we have

$$\begin{aligned}
|\dot{X}_i(t_2)^2 - \dot{X}_i(t_1)^2| &= \left|2\int_{t_1}^{t_2} \dot{X}_i(s)\ddot{X}_i(s)\,ds\right| \\
&\le C\int_{t_1}^{t_2} \mathcal{G}(|X(s)|)\dot{X}_i(s)\,ds \\
&\le C\int_{t_1}^{t_2} \mathcal{G}(|X_i(s)|)\dot{X}_i(s)\,ds \\
&=: C^{(0)}\int_{X_i(t_1)}^{X_i(t_2)} \mathcal{G}(|x|)\,dx \\
&\le C^{(0)}B(X_i(t_2) - X_i(t_1))^{1/b} + 10C^{(0)} \\
&\le C^{(0)}B((\sup_{\tau\in[t_1,t_2]} |\dot{X}_i(\tau)|)(t_2 - t_1))^{1/b} + 10C^{(0)}.
\end{aligned}$$

Define

$$W := \sup_{s\in[0,T]} |\dot{X}_i(s)|.$$

Then,

$$|\dot{X}_i(t_2)^2 - \dot{X}_i(t_1)^2| \le C^{(0)}B(Wt_2)^{1/b} + 10C^{(0)}. \tag{14}$$

Note that this holds if $\dot{X}_i \le 0$ on $[t_1,\ t_2]$, as well. Let us consider $t \in [0,\ T]$ and $\dot{X}_i(t) > 0$. Define

$$\bar{t} := \inf\{\tau \ge 0 : \dot{X}_i(s) \ge 0,\ \ \forall s \in [\tau, t]\}.$$

If $\bar{t} = 0$, then by (14)

$$\begin{aligned}
\dot{X}_i^2(t) &\le \dot{X}_i^2(0) + C^{(0)}B(Wt)^{1/b} + 10C^{(0)} \\
&\le \dot{X}_i^2(0) + C^{(0)}B(WT)^{1/b} + 10C^{(0)}.
\end{aligned}$$

If $\bar{t} > 0$, then $\dot{X}_i(\bar{t}) = 0$, and by (14), we have

$$\begin{aligned}\dot{X}_i^2(t) &\leq \dot{X}_i^2(\bar{t}) + C^{(0)}B(Wt)^{1/b} + 10C^{(0)}\\ &= C^{(0)}B(WT)^{1/b} + 10C^{(0)}\\ &\leq |\dot{X}_i^2(0)| + C^{(0)}B(WT)^{1/b} + 10C^{(0)}.\end{aligned}$$

We may repeat this process for $\dot{X}_i(t) < 0$, so summing over i yields

$$|\dot{X}(t)|^2 \leq |\dot{X}(0)|^2 + 3C^{(0)}B(WT)^{1/b} + 30C^{(0)}.$$

Since the right-hand side is independent of t, we take the supremum and find

$$W^2 \leq 3C^{(0)}BT^{1/b}W^{1/b} + M$$

where $M : = |\dot{X}(0)|^2 + 30\mathrm{C}^{(0)}$. If $3C^{(0)}BT^{1/b}W^{1/b} \leq M$, then

$$W \leq \sqrt{2M} =: \mathcal{C}_1(T).$$

If $M \leq 3C^{(0)}BT^{1/b}W^{1/b}$, then

$$W \leq (6C^{(0)}B)^{b/2b-1}T^{1/2b-1} := \mathcal{C}_2(T).$$

So, combining the inequalities,

$$W \leq \max\{\mathcal{C}_1(T), \mathcal{C}_2(T)\}.$$

Thus, all characteristics $V(s, t, x, v)$, along which f is nonzero, are bounded for any $s \in [0, T]$, including $V(t, t, x, v) = v$. Define

$$Q_t := \sup\{|v| : \exists x \in \mathbb{R}^3,\ \ \tau \in [0, t]\,\text{such that}\, f(\tau, x, v) \neq 0\}.$$

Notice that Q is an increasing function of time, so that we may write for every $s \in [0, T]$

$$|V(s, t, x, v)| \leq Q_s \leq Q_T.$$

Since the momentum is bounded, bounds on position follow. Note that

$$\begin{aligned}|X(s) - x| &\leq \int_s^t |V(\tau)|d\tau\\ &\leq Q_T T.\end{aligned}$$

So, for $|x| \geq 2Q_T T$,

$$|X(s)| \geq |x| - Q_T T \geq \frac{1}{2}|x| \tag{15}$$

and

$$|X(s)| \leq |x| + Q_T T \leq \frac{3}{2}|x|. \tag{16}$$

Hence, we have control on X characteristics.

Now, to bound the charge, we must first estimate the corresponding density. We use (2) to find

$$\frac{d}{ds}(g(s, X(s), V(s))) = -(E(s, X(s)) + A(s, X(s))) \cdot \nabla_v F(V(s)).$$

Thus,

$$g(t, x, v) = g(0, X(0), V(0)) - \int_0^t (E(\tau, X(\tau)) + A(\tau, X(\tau)))$$
$$\cdot \nabla_v F(V(\tau)) d\tau, \tag{17}$$

and by (III), (12), and (15),

$$\begin{aligned}
|g(t,x,v)| &\leq |g_0(X(0,t,x,v), V(0,t,x,v))| \\
&\quad + \int_0^t (|E(\tau, X(\tau,t,x,v))| + |A(\tau, X(\tau,t,x,v))|) \\
&\quad |\nabla_v F(V(\tau,t,x,v))| d\tau \\
&\leq C|X(0,t,x,v)|^{-2} + C\int_0^t |X(\tau,t,x,v)|^{-2} \; \|\nabla_v F\|_\infty d\tau \qquad (18) \\
&\quad + C\int_0^t |m(\tau, X(\tau,t,x,v))| \; |X(\tau,t,x,v)|^{-2} \|\nabla_v F\|_\infty d\tau \\
&\leq C|x|^{-2}\left(1 + \int_0^t |m(\tau, X(\tau,t,x,v))| d\tau\right)
\end{aligned}$$

for $|x| \geq 2Q_T T$.

To show that g decays like r^{-2}, we must bound the enclosed charge m. For $|x| \leq 2Q_T T$,

$$\begin{aligned}|m(t, |x|)| &\leq \int_{|y|\leq 2Q_T T} |\rho(t, y)| dy \\ &\leq (2Q_T T)^3 \int_{|v|\leq Q_T} \|F - f_0\|_\infty dv \\ &\leq C.\end{aligned}$$

Now, let $r = |x|$ and define $\mathscr{M}(t) := \sup_r |m(t, r)|$. We know by (2), (18), and the divergence theorem that

$$\begin{aligned}|m(t, r)| \leq |m(0, r)| &+ \left| \int_0^t \int_{|y|\leq r} \int_{|v|\leq Q_T} \partial_s(g(s, y, v)) dv dy ds \right| \\ &= |m(0, r)| + \left| \int_0^t \int_{|y|\leq r} \int_{|v|\leq Q_T} v \cdot \nabla_y g(s, y, v) dv dy ds \right| \\ &\leq |m(0, r)| + \int_0^t \int_{|y|=r} \int_{|v|\leq Q_T} |v| \; |g(s, |y|, |v|, y \cdot v)| dv dS_y ds \\ &\leq M(0) + C \int_0^t \int_{|y|=r} \int_{|v|\leq Q_T} |v| \left(\frac{1 + \int_0^s M(\tau) d\tau}{|y|^2} \right) dv dS_y ds \\ &= M(0) + C \int_0^t Q_T^4 (1 + \int_0^s M(\tau) d\tau) ds \\ &\leq C + C \int_0^t M(\tau) d\tau\end{aligned}$$

for $r \geq 2Q_T T$. Then, since $m(t, r) \leq C$ for $r \leq 2Q_T T$, we find, for all r,

$$m(t, r) \leq C + C \int_0^t M(\tau) d\tau,$$

and consequently

$$M(t) \leq C + C \int_0^t M(\tau) d\tau.$$

By Gronwall's inequality,

$$\begin{aligned}\mathcal{M}(t) &\le (\mathcal{M}(0) + CT)(1 + Ct\exp(Ct)) \\ &\le (\mathcal{M}(0) + CT)(1 + CT\exp(CT)) \le C. \end{aligned} \tag{19}$$

Notice, then, that this bound and (18) imply

$$|g(t,x,v)| \le \frac{C}{|x|^2}$$

for $|x| \ge 2Q_T T$.

Proceeding in the standard way, we next estimate the gradient of the electric field.

$$\begin{aligned}\partial_{x_i} E_k &= \frac{\partial}{\partial x_i}\left(\frac{x_k}{|x|^3}\int_{|y|\le|x|}\rho(t,y)dy\right) \\ &= \frac{\partial}{\partial x_i}\left(\frac{x_k}{|x|^3}\right)m(t,|x|) + \frac{x_k}{|x|^3}\frac{\partial}{\partial x_i}(m(t,|x|)).\end{aligned}$$

Then,

$$\left|\frac{\partial}{\partial x_i}\left(\frac{x_k}{|x|^3}\right)\right| = \left|\frac{\delta_{ik}}{|x|^3} - \frac{3x_i x_k}{|x|^5}\right| \le \frac{C}{|x|^3}$$

and, letting $r = |x|$,

$$\left|\frac{\partial}{\partial x_i} m(t,r)\right| = \left|m_r(t,r)\left(\frac{x_i}{r}\right)\right| \le C|\rho(t,r)|r^2.$$

Thus,

$$\begin{aligned}|\partial_{x_i} E_k| &\le |m(t,r)|\frac{C}{r^3} + \frac{C|\rho(t,r)|\ r^2}{r^2} \\ &\le \frac{C}{r^3} + C|\rho(t,r)|.\end{aligned}$$

Since the best we can do a priori is $|\rho(t,r)| \le Cr^{-2}$, we can only conclude

$$|\partial_{x_i} E_k| \le C|x|^{-2}. \tag{20}$$

We begin to estimate the spatial decay of ρ by first estimating the large $|x|$ behavior of the field integral obtained by integrating (17) in v. Assume $|x| \geq 8Q_T T$ and $f(t, x, v)$ is nonzero. Define

$$\mathcal{E}(t,x) = E(t,x) + A(t,x).$$

Then,

$$\begin{aligned}
&\left| \int_{|v|\leq Q_T} \mathcal{E}(s, X(s)) \cdot \nabla_v F(V(s)) dv \right| \\
&\leq \left| \int_{|v|\leq Q_T} \mathcal{E}(s, X(s)) \cdot (\nabla_v F(V(s)) - \nabla_v F(v)) dv \right| \\
&+ \left| \int_{|v|\leq Q_T} (\mathcal{E}(s, X(s)) - \mathcal{E}(s, x + (s-t)v)) \cdot \nabla_v F(v) dv \right| \\
&+ \left| \int_{|v|\leq Q_T} \nabla_v \cdot (F(v)\ \mathcal{E}(s, x + (s-t)v)) dv \right| \\
&+ \left| \int_{|v|\leq Q_T} F(v) \nabla_v \cdot (\mathcal{E}(s, x + (s-t)v)) dv \right| \\
&=: (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}).
\end{aligned}$$

By the mean value theorem, (12), (15), and (19),

$$\begin{aligned}
(I) &\leq \int_{|v|\leq Q_T} |\mathcal{E}(s, X(s))|\ \|\nabla_v^2 F\|_\infty\ |V(s) - v| dv \\
&\leq \int_{|v|\leq Q_T} (C|X(s)|^{-2} \|\nabla_v^2 F\|_\infty\ |\int_s^t E(\tau, X(\tau)) d\tau|) dv \qquad (21) \\
&\leq CQ_T^3 |x|^{-2} (t-s)|x|^{-2} \\
&\leq C|x|^{-4}.
\end{aligned}$$

To estimate (II), we again use the mean value theorem and (20), so that there is ξ_x between $X(s)$ and $x + (s-t)v$ with

$$\begin{aligned}
|\mathcal{E}_i(s, X(s)) - \mathcal{E}_i(s, x + (s-t)v)| &= |\nabla_x \mathcal{E}_i(s, \xi_x) \cdot (X(s) - x - (s-t)v)| \\
&\leq C|\xi_x|^{-2} |X(s) - x + (t-s)v|.
\end{aligned}$$

Thus, we find

$$\begin{aligned}(II) &\leq C\int_{|v|\leq Q_T} |\xi_x|^{-2}\ |X(s)-(x+(s-t)v)|\ \|\nabla_v F\|_\infty dv\\ &\leq C\int_{|v|\leq Q_T} |\xi_x|^{-2}\left(\int_s^t |V(\tau)-v|d\tau\right)dv\\ &\leq CT^2Q_T^3|x|^{-2}|\xi_x|^{-2},\end{aligned}$$

and, by (15),

$$\begin{aligned}|\xi_x| &\geq |X(s)|-|X(s)-(x+(s-t)v)|\\ &\geq \frac{1}{2}|x|-\left|\int_s^t (V(\tau)-v)d\tau\right|\\ &\geq \frac{1}{2}|x|-2TQ_T\\ &\geq \frac{1}{4}|x|.\end{aligned}$$

Therefore,

$$(II)\leq C|x|^{-4}. \tag{22}$$

By the divergence theorem, we find

$$(III)=0. \tag{23}$$

Then, evaluating IV yields

$$\begin{aligned}(IV) &\leq \int_{|v|\leq Q_T} \|F\|_\infty\ |\nabla_x\cdot\mathcal{E}(s,x+(s-t)v)|\ |s-t|dv\\ &\leq CT\int_{|v|\leq Q_T} |\rho(s,x+(s-t)v)+\nabla_x\cdot A(s,x+(s-t)v)|dv\\ &\leq C\int_{|v|\leq Q_T} |\rho(s,x+(s-t)v)|dv + CR^{-4}(x+(s-t)v).\end{aligned}$$

Since

$$|x+(s-t)v|\geq |x|-TQ_T\geq \frac{1}{2}|x|,$$

we have

$$(IV) \leq C \int_{|v| \leq Q_T} |\rho(s, x + (s-t)v)| dv + C|x|^{-4}. \tag{24}$$

Finally, collecting (21), (22), (23), and (24), we have for $|x| \geq 8Q_T T$,

$$\begin{aligned} \left| \int_{|v| \leq Q_T} \mathcal{E}(s, X(s)) \cdot \nabla_v F(V(s)) dv \right| &\leq C|x|^{-4} \\ &+ C \int_{|v| \leq Q_T} |\rho(s, x + (s-t)v)| dv. \end{aligned} \tag{25}$$

Now, we can bound $\|\rho(t)\|_4$. Using the bound on the velocity support, we have

$$\begin{aligned} \|\rho(t)\|_\infty &= \sup_x \left| \int_{|v| \leq Q_T} (F(v) - f(t, x, v)) dv \right| \\ &\leq \frac{4\pi}{3} Q_T^3 \, \|F - f_0\|_\infty \\ &\leq C. \end{aligned}$$

Thus, for $|x| < 8Q_T T$, we have

$$|\rho(t, x)| \leq \|\rho(t)\|_\infty \left(\frac{8Q_T T}{|x|} \right)^4 \leq C|x|^{-4}.$$

Now, recall (17):

$$\begin{aligned} g(t, x, v) = g(0, X(0, t, x, v), V(0, t, x, v)) - \int_0^t \mathcal{E}(s, X(s, t, x, v)) \\ \cdot \nabla_v F(V(s, t, x, v)) ds. \end{aligned}$$

We use this equation and (25) so that for $|x| \geq 8Q_T T$,

$$
\begin{aligned}
|\rho(t,x)| &= \left| \int_{|v|\leq Q_T} g(t,x,v)dv \right| \\
&\leq \int_{|v|\leq Q_T} \left(\left| g_0(X(0), V(0)) \right| \right. \\
&\quad \left. + \left| \int_0^t \mathcal{E}(s, X(s)) \cdot \nabla_v F(V(s))ds \right| \right) dv \\
&\leq CQ_T^3 |x|^{-4} + \int_0^t \left(CQ_T^3 |x|^{-4} + C \int_{|v|\leq Q_T} |\rho(s, x \right. \\
&\quad \left. + (s-t)v)|dv \right) ds \\
&\leq C|x|^{-4} + C \int_0^t \int_{|v|\leq Q_T} |\rho(s, x + (s-t)v)| dv\, ds.
\end{aligned}
$$

So,

$$
|x|^4 |\rho(t,x)| \leq C + C \int_0^t \int_{|v|\leq Q_T} |x|^4 |\rho(s, x + (s-t)v)| dv\, ds.
$$

Define

$$
\mathcal{P}(t) := \sup_x (|x|^4 |\rho(t,x)|).
$$

Then, for all $x \in \mathbb{R}^3$,

$$
\begin{aligned}
\mathcal{P}(t) &\leq C + C \int_{|v|\leq Q_T} \int_0^t \mathcal{P}(s) ds\, dv \\
&\leq C + C \int_0^t \mathcal{P}(s) ds,
\end{aligned}
$$

and using the Gronwall inequality, we find

$$
\mathcal{P}(t) \leq C,
$$

and thus $|\rho(t, x)| \leq C|x|^{-4}$ for all $x \in \mathbb{R}^3$. Finally, we may apply Theorem 2 with any $q > 7 + \sqrt{33}$, and since $\|\rho(t)\|_\infty$ is bounded, we conclude that $|\rho(t, x)| \leq CR^{-4}(x)$. Since this estimate is independent of T, we find

$$\sup_{t\in[0,T]} \|\rho(t)\|_p \leq C.$$

This remains true for any $T > 0$, so the proof of Lemma 1, and thus Theorem 1, is complete.

4. Increased Spatial Decay

As in Section 3, we will show that the charge density decays at a faster rate than previously known. In the work that follows, we will use the framework of the previous sections and assume conditions (I) and (II) from Section 3, without the spherical symmetry of f_0. However, we will not take condition (III) as an assumption, and instead, assume condition (IV) holds for some $C > 0$ and all $t \geq 0$, $x \in \mathbb{R}^3$, and $v \in \mathbb{R}^3$. Since we have made a change in the assumptions, we may not use results from the previous sections, unless otherwise stated. Thus, this section can be viewed independently from the others, as we will rely more on results shown previously by Schaeffer (2003b). Recall from the introduction that we will use $C_{p,t}$ to denote a generic constant that depends upon $\|\rho(t)\|_p$ and t.

To begin, we apply Theorems 1 and 2 of Schaeffer (2003b), finding a unique $f \in C^1([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3)$ that satisfies (1), and assume

$$|||(F - f)(t)||| < \infty \tag{26}$$

for all $t \in [0, T)$. To first bound the velocity support, we write, as before,

$$g(t, x, v) := F(v) - f(t, x, v)$$

and

$$\mathcal{E}(t, x) := E(t, x) + A(t, x).$$

Using Lemma 1 of Schaeffer (2003b), we find for all $x \in \mathbb{R}^3$ and $t \in [0, T)$,

$$\int_0^t |\mathcal{E}(\tau, x)| d\tau \leq C \sup_{\tau \in [0,t]} \|\rho(\tau)\|_p.$$

Then, for any $s \in [0, t)$,

$$\begin{aligned} |V(s) - V(0)| &\leq \int_0^s |\mathcal{E}(\tau, X(\tau, t, x, v))| d\tau \\ &\leq C_{p,s}. \end{aligned}$$

Assuming $f \neq 0$, we know $f(t, x, v) = f_0(X(0), V(0))$, and by (II) it must follow that

$$|V(0, t, x, v)| \leq C.$$

Thus, $|V(s)| \leq C_{p,\,s}$ for any $s \in [0, t)$. Following previous notation, let us write

$$Q_t := \sup\{|v| : \exists x \in \mathbb{R}^3,\ \tau \in [0, t) \text{ such that } f(\tau, x, v) \neq 0\}$$

so that for all $s \in [0, t)$,

$$|V(s)| \leq Q_t. \tag{27}$$

This bound on the velocity establishes some of the relations shown in previous sections. Most importantly, (15) and (16) must hold for $|x| \geq 2Q_t t$.

Next, we denote the v-derivatives of characteristics by

$$\mathbb{A}_{ij} := \frac{\partial X_i}{\partial v_j}(s, t, x, v)$$

and

$$\mathbb{B}_{ij} := \frac{\partial V_i}{\partial v_j}(s, t, x, v).$$

Then, using the characteristic equations of (28),

$$\left.\begin{array}{c}\dfrac{\partial \mathbb{A}}{\partial s}(s) = \mathbb{B}(s),\\ \dfrac{\partial \mathbb{B}}{\partial s}(s) = \nabla_x \mathcal{E}(s, X(s))\mathbb{A}(s),\\ \mathbb{A}|_{s=t} = 0,\\ \mathbb{B}|_{s=t} = \mathbb{I}.\end{array}\right\} \tag{28}$$

Thus,

$$|\mathbb{A}(s)| \leq \int_s^t |\mathbb{B}(\tau)| d\tau$$

and

$$|\mathbb{B}(s)| \leq 1 + \int_s^t |\nabla_x \mathcal{E}(\tau, X(\tau))||\mathbb{A}(\tau)| d\tau,$$

which leads to

$$|\mathbb{A}(s)| + |\mathbb{B}(s)| \leq 1 + \int_s^t (|\mathbb{B}(\tau)| + |\nabla_x \mathcal{E}(\tau, X(\tau))||\mathbb{A}(\tau)|) d\tau. \tag{29}$$

Again, using Lemma 1 from Schaeffer (2003b), we find

$$|\nabla_x \mathcal{E}(\tau, X(\tau))| \leq C_{p,\tau} R^{-3}(X(\tau)). \tag{30}$$

So, define

$$\mathcal{H}(s) := \sup_{\{(x,v):|x|>2Q_t t\}} (|\mathbb{A}(s, t, x, v)| + |\mathbb{B}(s, t, x, v)|). \tag{31}$$

Then, for $|x| > 2Q_t t$, we use (15) to find

$$R^{-3}(X(\tau)) \leq R^{-3}(Q_t t),$$

and thus, from (29),

$$\mathcal{H}(s) \leq 1 + \int_s^t \max\{1, C_{p,\tau} R^{-3}(Q_t t)\} \mathcal{H}(\tau) d\tau.$$

Using the Gronwall inequality, we find

$$\mathcal{H}(s) \leq C_{p,t} \tag{32}$$

for $s \in [0, t]$, and v-derivatives of both characteristics are bounded for $|x| > 2Q_t t$.

Summarizing (28), we may write

$$\frac{\partial^2 \mathbb{A}}{ds^2}(s) = \nabla_x \mathcal{E}(s, X(s))\mathbb{A}(s); \quad \mathbb{A}|_{s=t} = 0; \quad \frac{\partial \mathbb{A}}{\partial s}|_{s=t} = \mathbb{I}.$$

So,

$$\begin{aligned}\mathbb{A}(s) &= (s-t)\mathbb{I} + \int_s^t \int_\tau^t \nabla_x \mathcal{E}(\lambda, X(\lambda))\mathbb{A}(\lambda) d\lambda d\tau \\ &=: (s-t)\mathbb{I} + \gamma_1(s, t, x, v).\end{aligned}$$

Again using (28),

$$\begin{aligned}\frac{\partial \mathbb{B}}{\partial s}(s) &= \nabla_x \mathcal{E}(s, X(s))\mathbb{A}(s) \\ &= \nabla_x \mathcal{E}(s, X(s))((s-t)\mathbb{I} + \gamma_1(s, t, x, v)) \\ &=: (s-t)\nabla_x \mathcal{E}(s, X(s)) + \gamma_2(s, t, x, v).\end{aligned}$$

Thus,

$$\mathbb{B}(s) = \mathbb{I} + \gamma_3(s, t, x, v)$$

where

$$\gamma_3(s, t, x, v) = -\int_s^t ((\tau - t)\nabla_x \mathcal{E}(\tau, X(\tau)) + \gamma_2(\tau, t, x, v))d\tau.$$

Now,

$$\begin{aligned}\frac{\partial}{\partial s}(\det \mathbb{B}) &= \det\begin{pmatrix} \dot{\mathbb{B}}_{11} & \dot{\mathbb{B}}_{12} & \dot{\mathbb{B}}_{13} \\ \mathbb{B}_{21} & \mathbb{B}_{22} & \mathbb{B}_{23} \\ \mathbb{B}_{31} & \mathbb{B}_{32} & \mathbb{B}_{33} \end{pmatrix} + \det\begin{pmatrix} \mathbb{B}_{11} & \mathbb{B}_{12} & \mathbb{B}_{13} \\ \dot{\mathbb{B}}_{21} & \dot{\mathbb{B}}_{22} & \dot{\mathbb{B}}_{23} \\ \mathbb{B}_{31} & \mathbb{B}_{32} & \mathbb{B}_{33} \end{pmatrix} \\ &\quad + \det\begin{pmatrix} \mathbb{B}_{11} & \mathbb{B}_{12} & \mathbb{B}_{13} \\ \mathbb{B}_{21} & \mathbb{B}_{22} & \mathbb{B}_{23} \\ \dot{\mathbb{B}}_{31} & \dot{\mathbb{B}}_{32} & \dot{\mathbb{B}}_{33} \end{pmatrix} \\ &=: (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}).\end{aligned}$$

Estimating the first term yields

$$\begin{aligned}
(\mathrm{I}) &= \det\begin{pmatrix} (s-t)\frac{\partial\mathcal{E}_1}{\partial x_1}+(\gamma_2)_{11} & (s-t)\frac{\partial\mathcal{E}_1}{\partial x_2}+(\gamma_2)_{12} & (s-t)\frac{\partial\mathcal{E}_1}{\partial x_3}+(\gamma_2)_{13} \\ (\gamma_3)_{21} & 1+(\gamma_3)_{22} & (\gamma_3)_{23} \\ (\gamma_3)_{31} & (\gamma_3)_{32} & 1+(\gamma_3)_{33} \end{pmatrix} \\
&=: (s-t)\frac{\partial\mathcal{E}_1}{\partial x_1}+(\gamma_2)_{11}+\sigma_1(s,t,x,v).
\end{aligned}$$

Similarly,

$$(\mathrm{II}) =: (s-t)\frac{\partial\mathcal{E}_2}{\partial x_2}+(\gamma_2)_{22}+\sigma_2(s,t,x,v)$$

and

$$(\mathrm{III}) =: (s-t)\frac{\partial\mathcal{E}_3}{\partial x_3}+(\gamma_2)_{33}+\sigma_3(s,t,x,v).$$

So, we find

$$\frac{\partial}{\partial s}(\det\mathbb{B}) = (s-t)\nabla_x\cdot\mathcal{E}(s,x)|_{x=X(s)}+\sum_{j=1}^{3}((\gamma_2)_{jj}+\sigma_j),$$

and since $\mathbb{B}|_{s=t}=\mathbb{I}$,

$$\det\mathbb{B} = 1-4\pi\int_s^t(\tau-t)\rho(\tau,X(\tau))d\tau-\int_s^t\sum_{j=1}^{3}((\gamma_2)_{jj}+\sigma_j)d\tau.$$

Let

$$\varepsilon(s,t,x,v) := 4\pi\int_s^t(\tau-t)\rho(\tau,X(\tau))d\tau+\int_s^t\sum_{j=1}^{3}((\gamma_2)_{jj}+\sigma_j)d\tau.$$

Then, for $|\varepsilon|<1$,

$$\begin{aligned}
\frac{1}{\det\mathbb{B}} &= \frac{1}{1-\varepsilon} \\
&= 1+\varepsilon+\sum_{n=2}^{\infty}\varepsilon^n \qquad (33)\\
&=: 1+4\pi\int_s^t(\tau-t)\rho(\tau,X(\tau))d\tau+\eta(s,t,x,v).
\end{aligned}$$

Now that the determinant has been written in a nicer form, we estimate the remaining terms. Let $|x| > 2Q_t t$. Using (15), (26), (30), (31), and (32), we find the following bounds on the error terms for any $i, j = 1, 2, 3$:

$$\begin{aligned} |(\gamma_1)_{ij}| &\leq C_{p,t} T^2 R^{-3}(x) \sup_{s\in[0,t]} |\mathbb{A}(s,t,x,v)| \\ &\leq C_{p,t} R^{-3}(x), \\ |(\gamma_2)_{ij}| &\leq C_{p,t} R^{-6}(x), \end{aligned}$$

and

$$\begin{aligned} |(\gamma_3)_{ij}| &\leq C_{p,t} T^2 R^{-3}(x) + C_{p,t} R^{-6}(x) \\ &\leq C_{p,t} R^{-3}(x). \end{aligned}$$

Then, using the estimates of the γ terms, for any $k = 1, 2, 3$,

$$\begin{aligned} |\sigma_k| &\leq C(|\nabla_x \mathcal{E}| + |\gamma_2|)(2|\gamma_3| + 2|\gamma_3|^2) + 2(|\nabla_x \mathcal{E}| + |\gamma_2|)(|\gamma_3| + 2|\gamma_3|^2) \\ &\leq [C_{p,t} T R^{-3}(x) + C_{p,t} R^{-6}(x)][2C_{p,t} R^{-3}(x) + 2C_{p,t} R^{-6}(x)] \\ &\quad + 2[C_{p,t} T R^{-3}(x) + C_{p,t} R^{-6}][C_{p,t} R^{-3}(x) + 2C_{p,t} R^{-6}(x)] \\ &\leq C_{p,t} R^{-6}(x). \end{aligned}$$

Now, we may bound ε:

$$\begin{aligned} |\varepsilon| &\leq 4\pi T^2 \sup_{\tau\in[0,t]} \|\rho(\tau)\|_p R^{-p}(x) + 3T \sup_{\tau\in[s,t]} (|\gamma_2(\tau)| + |\sigma(\tau)|) \\ &\leq C_{p,t} R^{-p}(x) + C_{p,t} R^{-6}(x) \\ &\leq C_{p,t} R^{-p}(x) \\ &=: C^{(1)} R^{-p}(x) \\ &< \frac{1}{2} \end{aligned}$$

for $|x| > (2C^{(1)})^{1/p}$. Thus $\sum_{n=2}^{\infty} \varepsilon^n$ converges and η is well defined for large enough values of $|x|$. Finally, we have, for $|x| > \max\{2Q_t t,$

$(2C^{(1)})^{1/p}\}$,

$$\begin{aligned} |\eta| &= \left| \int_s^t \sum_{j=1}^3 ((\gamma_2)_{jj} + \sigma_j) d\tau + \sum_{n=2}^\infty \varepsilon^n \right| \\ &\leq 3C_{p,t} T R^{-6}(x) + ((C^{(1)})^2 R^{-2p}(x) + (C^{(1)})^3 R^{-3p}(x) + \ldots) \\ &\leq C_{p,t} R^{-6}(x). \end{aligned} \tag{34}$$

Using (27), we find for $t \in [0, T)$,

$$\begin{aligned} \|\rho(t)\|_\infty &= \sup_x \left| \int (F(v) - f(t, x, v)) dv \right| \\ &\leq \frac{4\pi}{3} Q_t^3 \|F - f_0\|_\infty \\ &\leq C_{p,t}. \end{aligned}$$

Thus, if $|x| \leq D$ for some $D > 0$, we have

$$\begin{aligned} |\rho(t, x)| &\leq \|\rho(t)\|_\infty R^6(D) R^{-6}(x) \\ &\leq C_{p,t} R^{-6}(x). \end{aligned} \tag{35}$$

Now, denote $C^{(2)} := \max\{2Q_t t, (2C^{(1)})^{1/p}, 2N\}$, and let $|x| > C^{(2)}$. Then, by (15) we have

$$|X(0, t, x, v)| \geq \frac{1}{2}|x| > N.$$

Using (II), (11), (33), and (34) to estimate $\rho(t, x)$ yields

$$\begin{aligned} \int f(t, x, v) dv &= \int f_0(X(0, t, x, v), V(0, t, x, v)) dv \\ &= \int F(V(0, t, x, v)) dv \\ &= \int F(V(0, t, x, v)) (\det \mathbb{B}(0)) \frac{1}{\det \mathbb{B}(0)} dv \\ &= \int F(V(0, t, x, v)) \Big(1 + 4\pi \int_0^t (\tau - t) \rho(\tau, X(\tau, t, x, v)) d\tau \\ &\quad + \eta(0, t, x, v)\Big) \Big(\det \frac{\partial V}{\partial v}(0, t, x, v)\Big) dv \end{aligned}$$

$$= \int F(w)dw + \int F(V(0,t,x,v))\eta(0,t,x,v)$$

$$(\det \frac{\partial V}{\partial v}(0,t,x,v))dv + 4\pi \int F(V(0,t,x,v))$$

$$\int_0^t (\tau - t)\rho(\tau, X(\tau,t,x,v))d\tau(\det \frac{\partial V}{\partial v}(0,t,x,v))dv.$$

Since $\rho(t, x) = \int (F(v) - f(t, x, v))\, dv$, we have

$$\rho(t,x) = 4\pi \int F(V(0,t,x,v)) \int_0^t (t - \tau)\rho(\tau, X(\tau,t,x,v))d\tau$$

$$(\det \frac{\partial V}{\partial v}(0,t,x,v))dv - \int F(V(0,t,x,v))\eta(0,t,x,v)$$

$$(\det \frac{\partial V}{\partial v}(0,t,x,v))dv. \tag{36}$$

Now, let

$$\Psi(t) := \|\rho(t)\|_6 = \sup_x (|\rho(t,x)|R^6(x)).$$

In order to make the change of variables $w = V(0, t, x, v)$ in the v-integral, we must first show that the mapping $v \to V(0, t, x, v)$ is bijective. For the moment, we will take this for granted, and continue with the estimate of ρ, delaying the proof of this fact until the very end. By (34), (36), and the work of Section 5, we have for $|x| > C^{(2)}$,

$$|\rho(t,x)| \leq 4\pi \int F(V(0,t,x,v))(\int_0^t (t - \tau)\Psi(\tau)R^{-6}(X(\tau,t,x,v))d\tau)$$

$$(\det \frac{\partial V}{\partial v}(0,t,x,v))dv + C_{p,t}R^{-6}(x) \int F(w)dw$$

$$\leq CR^{-6}(x) \int F(V(0,t,x,v))(\det \frac{\partial V}{\partial v}(0,t,x,v))dv$$

$$\int_0^t (t - \tau)\Psi(\tau)d\tau + C_{p,t}R^{-6}(x)$$

$$\leq R^{-6}(x)(C \int_0^t (t - \tau)\Psi(\tau)d\tau + C_{p,t}). \tag{37}$$

So, applying (35) with $D = C^{(2)}$ and combining with (37), we have for all x,

$$|\rho(t,x)|R^6(x) \leq C_{p,t} + C\int_0^t (t-\tau)\Psi(\tau)d\tau$$

and since the right side is independent of $|x|$,

$$\Psi(t) \leq C_{p,t} + C\int_0^t (t-\tau)\Psi(\tau)d\tau.$$

By the Gronwall inequality,

$$\Psi(t) \leq C_{p,t}.$$

Therefore, for every $t \in [0, T)$,

$$\|\rho(t)\|_6 \leq C_{p,t}$$

and the proof is complete.

5. Change of Variables

In order to justify the change of variables used in (37), we must first demonstrate that the mapping $v \to V(0, t, x, v)$ is bijective. This is done below.

From (30), we know there is $C^{(3)}_{p,t} > 0$ such that

$$|\nabla\mathcal{E}(t,x)| \leq C^{(3)}_{p,t}|x|^{-3}$$

for every $x \in \mathbb{R}^3$, $t \in [0, T]$. Also, using (32), we know there is $C^{(4)}_{p,t} > 0$ such that for all s, $t \in [0, T]$, x, $v \in \mathbb{R}^3$, with $|x| > 2TQ(T)$,

$$\left|\frac{\partial X}{\partial v}(s,t,x,v)\right| \leq C^{(4)}_{p,t}.$$

Finally, using Lemma 1 of Schaeffer (2003b), we know there is $C^{(5)}_{p,t} > 0$ such that for all $x \in \mathbb{R}^3$ and $t \in [0, T]$,

$$\int_0^t |\mathcal{E}(\tau,x)|d\tau \leq C^{(5)}_{p,t}.$$

For any $D > 0$, define $C_D := \max\{8\,(D + C_{p,t}^{(5)}T), 2TQ(T), 4\,(6C_{p,t}^{(4)} C_{p,t}^{(3)}T)^{1/3}\}$, and $B(0, D) := \{v \in \mathbb{R}^3 : |v| \leq D\}$. Injectivity on $B(0, D)$ can now be shown in the following lemma.

Lemma 3. *For any* $D > 0$, $|x| > C_D$, *and* $t \in [0, T]$, *the mapping* $v \to V(0, t, x, v)$ *is injective on* $B(0, D)$.

Proof

Let $D > 0$, $|x| > C_D$ and $t \in [0, T]$ be given. Then, let v_1, $v_2 \in B(0, \mathrm{D})$ be given with

$$V(0, t, x, v_1) = V(0, t, x, v_2).$$

We have

$$v_1 + \int_0^t \mathcal{E}(\tau, X(\tau, t, x, v_1))d\tau = v_2 + \int_0^t \mathcal{E}(\tau, X(\tau, t, x, v_2))d\tau.$$

So, using the mean value theorem, for any $i = 1, 2, 3$ there is ξ_x^i between $X(\tau, t, x, v_1)$ and $X(\tau, t, x, v_2)$ and ξ_v^i between v_1 and v_2 such that

$$\begin{aligned}\mathcal{E}_i(\tau, X(\tau, t, x, v_1)) &- \mathcal{E}_i(\tau, X(\tau, t, x, v_2))\\ &= \nabla_x \mathcal{E}_i(\tau, \xi_x^i) \cdot (X(\tau, t, x, v_1) - X(\tau, t, x, v_2))\end{aligned}$$

and

$$X_i(\tau, t, x, v_1) - X_i(\tau, t, x, v_2) = \nabla_v X_i(\tau, t, x, \xi_v^i) \cdot (v_2 - v_1).$$

Then, since

$$\begin{aligned}|\xi_x^i| &\geq |X(\tau, t, x, v_1)| - |X(\tau, t, x, v_1) - X(\tau, t, x, v_2)|\\ &\geq \frac{1}{2}|x| - \int_\tau^t |V(\iota, t, x, v_1) - V(\iota, t, x, v_2)| d\iota\\ &\geq \frac{1}{2}|x| - \int_\tau^t (|V(\iota, t, x, v_1) - v_1| + |v_1 - v_2| + |V(\iota, t, x, v_2) - v_2|) d\iota\\ &\geq \frac{1}{2}|x| - \left(|v_1 - v_2| + 2\int_\tau^t \int_0^\iota \|E(\lambda)\|_\infty d\lambda\, d\iota\right)\end{aligned}$$

$$\geq \frac{1}{2}|x| - 2(D + C^{(5)}_{p,t}T)$$
$$\geq \frac{1}{4}|x|,$$

and

$$|X_i(\tau, t, x, v_1) - X_i(\tau, t, x, v_2)| \leq C^{(4)}_{p,t}|v_2 - v_1|,$$

for any $i = 1, 2, 3$, we find

$$\begin{aligned}|\mathcal{E}_i(\tau, X(\tau, t, x, v_1)) - \mathcal{E}_i(\tau, X(\tau, t, x, v_2))| &\leq |\nabla_x \mathcal{E}_i(\tau, \xi^i_x)|\ |X(\tau, t, x, v_1) \\ &\quad - X(\tau, t, x, v_2)| \\ &\leq \left(\sup_i C^{(3)}_{p,t}|\xi^i_x|^{-3}\right) \\ &\quad \left(\sqrt{3}C^{(4)}_{p,t}|v_1 - v_2|\right) \\ &\leq \sqrt{3}C^{(3)}_{p,t}C^{(4)}_{p,t}\left(\frac{1}{4}|x|\right)^{-3} \\ &\quad |v_1 - v_2|.\end{aligned}$$

Therefore, we have

$$\begin{aligned}|v_1 - v_2| &\leq \int_0^t |\mathcal{E}(\tau, X(\tau, t, x, v_1)) - \mathcal{E}(\tau, X(\tau, t, x, v_2))| d\tau \\ &\leq 3C^{(4)}_{p,t}|v_1 - v_2|C^{(3)}_{p,t}\int_0^t \left(\frac{1}{4}|x|\right)^{-3} d\tau. \\ &\leq \left(3C^{(3)}_{p,t}C^{(4)}_{p,t}T\left(\frac{1}{4}|x|\right)^{-3}\right)|v_1 - v_2| \\ &\leq \frac{1}{2}|v_1 - v_2|.\end{aligned}$$

Thus, $|v_1 - v_2| = 0$, which implies $v_1 = v_2$, and injectivity follows.

Now, let $t \in [0, T]$ and $x \in \mathbb{R}^3$ be given. Define

$$S := \{w : F(w) \neq 0\}$$

and

$$V^{-1}(S) := \{v : F(V(0, t, x, v)) \neq 0\}.$$

Using the compact support of F, we conclude that $v \in V^{-1}(S)$ implies

$$|v| \leq W + \int_0^t \|\mathcal{E}(\tau)\|_\infty d\tau \leq C_{p,t}.$$

Therefore, there is a $D > 0$, such that $V^{-1}(S) \subset B(0, D)$. Thus, for $|x| > C_D$, and $t \in [0, T]$, the mapping $v \to V(0, t, x, v)$ is injective on $V^{-1}(S)$ and bijective from $V^{-1}(S)$ to S. Finally,

$$\begin{aligned}\int F(V(0, t, x, v)) \det\left(\frac{\partial V}{\partial v}\right) dv &= \int_{V^{-1}(S)} F(V(0, t, x, v)) \det\left(\frac{\partial V}{\partial v}\right) dv\\ &= \int_S F(w) dw\\ &= \int F(w) dw.\end{aligned}$$

Thus, the change of variables is valid and the justification is complete.

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