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Global Existence for the Vlasov-Poisson System with Steady Spatial Asymptotics

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A collisionless plasma is modelled by the Vlasov-Poisson system in three space dimensions. A fixed background of positive charge—dependant upon only velocity—is assumed. The situation in which mobile negative ions balance the positive charge as $|x| \to \infty$ is considered. Thus, the total positive charge and the total negative charge are both infinite. Smooth solutions with appropriate asymptotic behavior for large $|x|$, which were previously shown to exist locally in time, are continued globally. This is done by showing that the charge density decays at least as fast as $|x|^{-6}$. This article also establishes decay estimates for the electrostatic field and its derivatives.

Keywords Cauchy problem; Global existence; Kinetic theory; Spatial decay; Vlasov.

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Introduction

Let $F : \mathbb{R}^3 \to [0, \infty)$, $f_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$, and $A : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ be given. We seek a solution, $f : [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ satisfying

$$\left.\begin{aligned} \partial_t f + v \cdot \nabla_x f - (E + A) \cdot \nabla_v f = 0, \\ \rho(t, x) = \int (F(v) - f(t, x, v))dv, \\ E(t, x) = \int \rho(t, y) \frac{x - y}{|x - y|^3} dy, \\ f(0, x, v) = f_0(x, v). \end{aligned}\right\} \tag{1}$$

Here F describes a number density of positive ions which form a fixed background, and f denotes the density of mobile negative ions in phase space. Notice that if $f_0(x, v) = F(v)$ and $A = 0$, then $f(t, x, v) = F(v)$ is a steady solution. Thus, we seek solutions for which $f(t, x, v) \to F(v)$ as $|x| \to \infty$. It is important to

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notice that (1) is a representative problem, and that problems concerning multiple species of ions can be treated in a similar manner.

Precise conditions which ensure local existence and conditions for continuation were given in Schaeffer (2003b). Also, *a priori* bounds on ρ and a quantity related to the energy were obtained in Schaeffer (2003a). Therefore, we will work towards establishing global existence with these bounds and using similar assumptions. Other work on the infinite mass case has been done by Jabin (2001) and Caglioti et al. (2001).

The case when $F(v) = 0$ and $f \to 0$ as $|x| \to \infty$ has been studied extensively. Smooth solutions were shown to exist globally in time in Pfaffelmoser (1992) and independently in Lions and Pertham (1991). Important results prior to global existence appear in Batt (1977), Glassey and Strauss (1986), Horst (1981, 1982), and Kurth (1952). Also, the method used by Pfaffelmoser (1992) is refined in Horst (1993) and Schaeffer (1991). Global existence for the Vlasov-Poisson system in two dimensions was established in Okabe and Ukai (1978) and Wollman (1980). A complete discussion of the literature concerning Vlasov-Poisson may be found in Glassey (1996). We also mention Batt and Rein (1991) and Rein and Rendall (1994) since the problem treated in these papers is periodic in space, and thus the solution does not decay for large $|x|$.

1. Section 1

Let $p \in (3, 4)$ and denote

$$R(x) = R(|x|) = (1 + |x|^2)^{\frac{1}{2}}.$$

We will use the notation

$$\|g\|_{L^\infty(\mathbb{R}^n)} = \sup_{z \in \mathbb{R}^n} |g(z)|$$

and

$$\|\rho\|_p := \|\rho(x) R^p(x)\|_{L^\infty(\mathbb{R}^3)},$$

but never use L^p. We will write, for example, $\|\rho(t)\|_p$ for the $\|\cdot\|_p$ norm of $x \mapsto \rho(t, x)$.

Following Schaeffer (2003a,b), we assume the following conditions hold for some $C > 0$ and all $t \geq 0$, $x \in \mathbb{R}^3$, and $v \in \mathbb{R}^3$, unless otherwise stated:

(I) $F(v) = F_R(|v|)$ is non-negative and C^2 with

$$F_R''(0) < 0, \tag{2}$$

and there is $W \in (0, \infty)$ such that

$$\left.\begin{array}{ll} F_R'(u) < 0 & \text{for } u \in (0, W) \\ F_R(u) = 0 & \text{for } u \geq W. \end{array}\right\} \tag{3}$$

(II) f_0 is C^1 and non-negative.

(III) A is C^1 with

$$|A(t, x)| \leq C^{(0)} R^{-2}(x), \tag{4}$$

$$|\partial_{x_i} A(t, x)| \leq C R^{-3}(x), \tag{5}$$

and

$$\nabla_x \cdot A(t, x) = 0. \tag{6}$$

Finally, we assume there is a continuous function $a : [0, T] \to \mathbb{R}$ such that

$$\left| A(t, x) - a(t) \frac{x}{|x|^3} \right| \leq CR^{1-p}(x).$$

(IV) $F - f_0$ has compact support in v, and there is $N > 0$ such that for $|x| > N$, we have

$$|F(v) - f_0(x, v)| \leq CR^{-6}(x).$$

Then, we have global existence.

Theorem 1. *Assuming conditions* (I)–(IV) *hold, there exists* $f \in \mathscr{C}^1([0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3)$ *that satisfies* (1) *with* $\| \int (F - f)(t) dv \|_p$ *bounded on* $t \in [0, T]$, *for every* $T > 0$. *Moreover,* f *is unique.*

In addition, due to the previously known result of Pankavich (2004), stated here as Theorem 2, we are able to conclude further decay of the charge density in Corollary 1.

Theorem 2. *Let* $T > 0$ *and* f *be the* $\mathscr{C}^1$ *solution of* (1) *on* $[0, T] \times \mathbb{R}^3 \times \mathbb{R}^3$. *Then, we have*

$$\|\rho(t)\|_6 \leq C_{p,t}$$

for any $t \in [0, T]$, *where* $C_{p,t}$ *depends upon*

$$\sup_{\tau \in [0,t]} \|\rho(\tau)\|_p.$$

Corollary 1. *Let* $T > 0$ *and* f *be the* $\mathscr{C}^1$ *solution of* (1) *on* $[0, T] \times \mathbb{R}^3 \times \mathbb{R}^3$. *Then, we have*

$$\left\| \int (F - f)(t) dv \right\|_6 \leq C$$

for any $t \in [0, T]$.

To prove Theorem 1, we will use Theorems 1 and 3 of Schaeffer (2003b), which guarantee local existence and continuation of the local solution so long as $\| \int (F - f)(t) \|_p$ is bounded for some $p > 3$. Therefore, the following lemma will be all that is needed to complete the proof of the theorem.

Lemma 1. *Assume conditions* (I)–(IV) *hold. Let* f *be a* $\mathscr{C}^1$ *solution of* (1) *on* $[0, T] \times \mathbb{R}^3 \times \mathbb{R}^3$. *Then,*

$$\left\| \int (F - f)(t) dv \right\|_p \leq C$$

for all $t \in [0, T]$, *where* C *is determined by* F, A, f_0, *and* T.

Define

$$Q_f(t) := \sup\{|v| : \exists x \in \mathbb{R}^3, \tau \in [0, t] \text{ such that } f(\tau, x, v) \neq 0\} \tag{7}$$

and

$$Q_g(t) := \max\{W, Q_f(t)\}.$$

In Section 2, we will bound E, ∇E, and $\nabla_{x,v} f$. Also, we will estimate the energy, and obtain bounds on $Q_f(t)$ and thus $Q_g(t)$. To better reveal the line of thought, the proofs of Lemmas 2 through 4 are deferred to Section 3.

We will denote by "C" a generic constant which changes from line to line and may depend upon f_0, A, F, or T, but not on t, x, or v. When it is necessary to refer to a specific constant, we will use numeric superscripts to distinguish them. For example, $C^{(0)}$, as in (III), will always refer to the same numerical constant.

To estimate E and ∇E, we will use the following lemmas.

Lemma 2. *For any $q > 0$ and $b \in [0, \frac{5}{18})$ with $b \leq \frac{2}{q}$, we have*

$$|E(t, x)| \leq C(\|\rho(t)\|_q R^{-q}(x))^b$$

for any $t \in [0, T]$, $|x| \geq 1$, where C may depend upon $\|\rho(t)\|_\infty$.

Lemma 3. *For any $q > 0$ and $a \in [0, 1)$ with $a \leq \frac{3}{q}$, we have*

$$|\nabla E(t, x)| \leq C(\|\rho(t)\|_q R^{-q}(x))^a$$

for any $t \in [0, T]$, $|x| \geq 1$, where C may depend upon $\|\rho(t)\|_\infty$ and $\|\nabla\rho(t)\|_\infty$.

Then, we will use the following lemma to bound the p-norm of ρ and obtain decay of both E and ∇E.

Lemma 4. *For any $q \in [0, \frac{54}{13})$, $\|\rho(t)\|_q$ is bounded for $t \in [0, T]$, where C may depend upon $\|\rho(t)\|_\infty$, $\|\nabla\rho(t)\|_\infty$, and $Q_f(T)$.*

Thus, once we show $\|\rho(t)\|_\infty$ and $\|\nabla\rho(t)\|_\infty$ are bounded for $t \in [0, T]$, and $Q_f(T)$ is finite, we may use Lemma 4 to bound $\|\rho(t)\|_p$ since $p \in (3, 4)$. Then, we can prove Lemma 1, and thus Theorem 1.

2. Section 2

2.1. *Characteristics*

Define the characteristics, $X(s, t, x, v)$ and $V(s, t, x, v)$, by

$$\left.\begin{array}{c} \dfrac{\partial X}{\partial s}(s, t, x, v) = V(s, t, x, v) \\ \dfrac{\partial V}{\partial s}(s, t, x, v) = -(E(s, X(s, t, x, v)) + A(s, X(s, t, x, v))) \\ X(t, t, x, v) = x \\ V(t, t, x, v) = v. \end{array}\right\} \tag{8}$$

Then, we have

$$\frac{\partial}{\partial s} f(s, X(s,t,x,v), V(s,t,x,v)) = \partial_t f + V \cdot \nabla_x f - (E+A) \cdot \nabla_v f = 0.$$

Therefore, f is constant along characteristics, and

$$f(t,x,v) = f(0, X(0,t,x,v), V(0,t,x,v)) = f_0(X(0,t,x,v), V(0,t,x,v)). \tag{9}$$

Thus, we find by (II) that f is non-negative.

Define $g : [0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by

$$g(t,x,v) := F(v) - f(t,x,v).$$

Then, we see that $\sup_{x,v} |g| \leq \|F\|_{L^\infty} + \|f_0\|_{L^\infty} < \infty$, and

$$\begin{aligned} \frac{\partial}{\partial s} g(s, X(s,t,x,v), V(s,t,x,v)) &= \partial_t g + V \cdot \nabla_x g - (E+A) \cdot \nabla_v g \\ &= -\nabla_v F(V(s)) \cdot (E+A)(s, X(s)). \end{aligned} \tag{10}$$

Finally, for any $s \in [0,T]$ with $f(s, X(s), V(s)) \neq 0$, we have for $t \in [0,T]$ and $x, v \in \mathbb{R}^3$,

$$|X(s,t,x,v) - x| = \left| \int_s^t \dot{X}(\tau,t,x,v)\,d\tau \right| \leq \int_s^t |V(\tau,t,x,v)|\,d\tau \leq TQ_g(T).$$

So, assuming we can bound $Q_g(T)$, we have for $|x| \geq 2TQ_g(T)$, $v \in \mathbb{R}^3$, $t \in [0,T]$, and $s \in [0,t]$,

$$|X(s,t,x,v)| \geq |x| - TQ_g(T) \geq \frac{1}{2}|x| \tag{11}$$

and

$$|X(s,t,x,v)| \leq |x| + TQ_g(T) \leq \frac{3}{2}|x|. \tag{12}$$

Unless it is necessary, we will omit writing the dependence of $X(s)$ and $V(s)$ on t, x, and v for the remainder of the article.

2.2. *Bounds on the Field*

Much of the work that follows will rely on energy estimates found in Schaeffer (2003a). In particular, we combine Lemma 3 and Theorem 4 from Schaeffer (2003a) to obtain an a priori bound on ρ.

First, define $\sigma : [0,\infty) \to \mathbb{R}$ by

$$\sigma(h) = -\int_0^{\min\{h, F(0)\}} \left(F^{-1}(\tilde{h})\right)^2 d\tilde{h}$$

and $S : [0, \infty) \times [0, \infty) \to [0, \infty)$ by

$$S(h, \eta) = (h - F(\eta))\eta^2 + \sigma(h) - \sigma(F(\eta)).$$

Then, from Schaeffer (2003a), we have the following lemma.

Lemma 5. *Let*

$$k(t, x) = \int S(f(t, x, v), |v|)dv.$$

Assuming conditions (I)–(IV) *hold, we have*

$$\int |F(v) - f(t, x, v)|dv \leq C(k(t, x)^{\frac{3}{5}} + k(t, x)^{\frac{1}{2}}) \tag{13}$$

and

$$\int k(t, x)dx \leq C. \tag{14}$$

It follows directly from (13) that

$$|\rho(t, x)| \leq C(k(t, x)^{\frac{3}{5}} + k(t, x)^{\frac{1}{2}}). \tag{15}$$

Then, using Lemma 5 , we may bound the field.

Assume $\|\rho(t)\|_{L^\infty(\mathbb{R}^6)} \leq C$ for all $t \in [0, T]$. Then, for $t \geq 0$, $x \in \mathbb{R}^3$, and any $R > 0$

$$\left.\begin{aligned}
|E(t, x)| &\leq \int \frac{|\rho(t, y)|}{|x - y|^2}dy \\
&\leq \int_{|x-y|<R} \|\rho(t)\|_{L^\infty}|x - y|^{-2}dy + \int_{|x-y|>R} |\rho(t, y)||x - y|^{-2}dy \\
&\leq 4\pi\|\rho(t)\|_{L^\infty} \int_0^R dr + C\int_{|x-y|>R} \left(k^{\frac{1}{2}}(t, y) + k^{\frac{3}{5}}(t, y)\right)|x - y|^{-2}dy \\
&\leq 4\pi\|\rho(t)\|_{L^\infty}R + C\Big(\int k(t, y)dy\Big)^{\frac{1}{2}}\Big(\int_{|x-y|>R} |x - y|^{-4}\Big)^{\frac{1}{2}} \\
&\quad + C\Big(\int k(t, y)dy\Big)^{\frac{3}{5}}\Big(\int_{|x-y|>R} |x - y|^{-5}\Big)^{\frac{2}{5}} \\
&\leq 4\pi\|\rho(t)\|_{L^\infty}R + C\Big(\int_R^\infty r^{-2}dr\Big)^{\frac{1}{2}} + C\Big(\int_R^\infty r^{-3}dr\Big)^{\frac{2}{5}} \\
&\leq C(\|\rho(t)\|_{L^\infty}R + R^{-\frac{1}{2}} + R^{-\frac{4}{5}}).
\end{aligned}\right\} \tag{16}$$

Obviously, we may choose $R = 1$ and deduce that $|E(t, x)| \leq C$. Suppose that the v-support of g is bounded for bounded times. Then, we find

$$|\rho(t, x)| \leq \int_{|v|\leq Q_g(t)} |g(t, x, v)|dv \leq (\|f_0\|_{L^\infty} + \|F\|_{L^\infty})Q_g^3(t) \leq CQ_g^3(t).$$

Thus, we know for every $t \in [0, T]$,

$$\|\rho(t)\|_{L^\infty} \leq CQ_g^3(t).$$

Now, choose $R = Q_g^{-\frac{5}{3}}(t)$. If $R \geq 1$, then $Q_g(t)$ is bounded. Otherwise, $R \leq 1$ and thus $R^{-\frac{1}{2}} \leq R^{-\frac{4}{5}}$.

Then, we have

$$\left.\begin{aligned} |E(t,x)| &\leq C\Big(\|\rho(t)\|_{L^\infty} R + R^{-\frac{4}{5}}\Big) \\ &\leq C\left(Q_g^3(t) Q_g^{-\frac{5}{3}}(t) + (Q_g^{-\frac{5}{3}}(t))^{-\frac{4}{5}}\right) \\ &\leq C Q_g^{\frac{4}{3}}(t) \\ &=: C^{(1)} Q_g^{\frac{4}{3}}(t). \end{aligned}\right\} \tag{17}$$

2.3. *Bounds on Derivatives of Density and Field*

We proceed as in Section 4.2.5 of Glassey (1996) with one modification: we will not assume that $\rho(t) \in L^1(\mathbb{R}^3)$ for all $t \in [0, T]$. Instead, we find for any $R > 0$, (letting $r = |x - y|$)

$$\begin{aligned} &\left|\frac{1}{4\pi}\int_{|y-x|\geq R} \rho(t,y)\left(\frac{3(y_k - x_k)^2}{r^5} - \frac{1}{r^3}\right) dy\right| \\ &\leq \frac{1}{\pi}\int_{|y-x|\geq R} |\rho(t,y)| r^{-3} dy \leq C\int_{|y-x|\geq R} \big(k^{\frac{1}{2}}(t,y) + k^{\frac{3}{5}}(t,y)\big) r^{-3} dy \\ &\leq C\bigg(\Big(\int k(t,y)dy\Big)^{\frac{1}{2}}\Big(\int_{|y-x|\geq R} r^{-6} dy\Big)^{\frac{1}{2}} + \Big(\int k(t,y)dy\Big)^{\frac{3}{5}}\Big(\int_{|y-x|\geq R} r^{-\frac{15}{2}} dy\Big)^{\frac{2}{5}}\bigg) \\ &\leq C\Big(R^{-\frac{3}{2}} + R^{-\frac{9}{5}}\Big). \end{aligned}$$

Thus, introducing this estimate into the result of Glassey (1996), we have for any $0 < d \leq R$

$$|\nabla_x E(t,x)| \leq C(1 + \ln(R/d))\|\rho(t)\|_{L^\infty} + Cd\|\nabla_x \rho(t)\|_{L^\infty} + C(R^{-\frac{3}{2}} + R^{-\frac{9}{5}}). \tag{18}$$

Then, we may follow the argument in Section 4.2.6 of Glassey (1996) and find a priori bounds on $\|\nabla_x E(t)\|_{L^\infty}$ and $\|\nabla_{x,v} f(t)\|_{L^\infty}$ as long as $\|\rho(t)\|_{L^\infty}$ is bounded. This work will be included in Appendix A.

2.4. *Energy Bound*

First, notice from (3) that we may write $W > 0$ as

$$W = \inf\{\eta > 0 : F(\eta) = 0\}.$$

Recall the definitions of σ, S, and k, as well as, (14), which we may write as

$$\iint S(f(t,x,v), |v|) dv\, dx \leq C$$

for any $t \in [0, T]$.

Let $\eta \geq 2W$. We find

$$\begin{aligned} S(h,\eta) &= (h - F(\eta))\eta^2 + \sigma(h) - \sigma(F(\eta)) = h\eta^2 + \sigma(h) - \sigma(0) = h\eta^2 + \sigma(h) \\ &= h\eta^2 - \int_0^{\min\{h,F(0)\}} (F^{-1}(\tilde{h}))^2 d\tilde{h} \geq h\eta^2 - \min\{h, F(0)\}(F^{-1}(0))^2 \\ &\geq h\eta^2 - hW^2 = h(\eta^2 - W^2). \end{aligned}$$

Then, since $\eta \geq 2W$, we have $\frac{1}{4}\eta^2 \geq W^2$ and

$$h(\eta^2 - W^2) = \frac{1}{2}h\eta^2 + h\left(\frac{1}{2}\eta^2 - W^2\right) \geq \frac{1}{2}h\eta^2.$$

Thus, for any $h \geq 0$ and $\eta \geq 2W$,

$$S(h,\eta) \geq \frac{1}{2}h\eta^2$$

and, finally, for $P \geq 2W$

$$\iint_{|v|>P} |v|^2 f(t,x,v)dv\,dx \leq 2\iint S(f(t,x,v),|v|)dv\,dx \leq C. \tag{19}$$

2.5. *The Good, The Bad, and The Ugly Revisited*

The method we will employ is very similar to that used in Section 4.4 of Glassey (1996). The differences are due mainly to the lack of positivity in the charge density, changes in the conserved energy, and the contribution of the applied field. We will assume throughout this section that $Q_g(t)$ is not already bounded and (17) applies.

Define

$$Q(t) := \left(\max\{(2W)^{\frac{4}{3}}, C^{(0)}\}\right)^{\frac{15}{13}} + Q_f(t).$$

Then, define $C^{(2)} := (C^{(0)})^{-\frac{7}{13}} + C^{(1)}$ so that we use (17) and (III) to find

$$\begin{aligned} |E(\tau,x) + A(\tau,x)| &\leq C^{(1)}Q^{\frac{4}{3}}(t) + C^{(0)}R^{-2}(x) \leq C^{(1)}Q^{\frac{4}{3}}(t) + \left(C^{(0)}\right)^{-\frac{7}{13}}\left(C^{(0)}\right)^{\frac{20}{13}} \\ &\leq C^{(1)}Q^{\frac{4}{3}}(t) + \left(C^{(0)}\right)^{-\frac{7}{13}}Q^{\frac{4}{3}}(t) = C^{(2)}Q^{\frac{4}{3}}(t) \end{aligned}$$

for any $\tau \in [0,t]$.

Now, let $(\widehat{X}(t), \widehat{V}(t))$ be any fixed characteristic

$$\frac{d}{dt}\widehat{X} = \widehat{V}, \quad \frac{d}{dt}\widehat{V} = E(t,\widehat{X})$$

for which

$$f(t,\widehat{X}(t),\widehat{V}(t)) \neq 0.$$

For any $0 \le \Delta \le t$, we have

$$\int_{t-\Delta}^{t} |E(s, \widehat{X}(s))| ds \le C \int_{t-\Delta}^{t} \iint \frac{|g(s, y, w)|}{|y - \widehat{X}(s)|^2} dw\, dy\, ds \tag{20}$$

$$= C \int_{t-\Delta}^{t} \iint \frac{|g(s, X(s, t, x, v), V(s, t, x, v))|}{|X(s, t, x, v) - \widehat{X}(s)|^2} dv\, dx\, ds. \tag{21}$$

Let $P = Q^{\frac{13}{20}}(t)$, $R > 0$, and $\Delta = \frac{P}{4C^{(2)} Q^{\frac{4}{3}}(t)}$. From the definition of Q, notice that $P \ge 2W$.

Let us partition the integral into I_G, I_B, and I_U, where I_A is the integral in (21) over the set A, and the three sets are defined as:

$$\begin{aligned} G &:= \left\{(s, x, v) : t - \Delta < s < t \text{ and } (|v| < P \text{ or } |v - \widehat{V}(t)| < P)\right\}, \\ B &:= \big\{(s, x, v) : t - \Delta < s < t \text{ and } |v| > P \text{ and } |v - \widehat{V}(t)| > P \\ &\qquad \text{and } |X(s, t, x, v) - \widehat{X}(s)| < R\big\}, \\ U &:= \big\{(s, x, v) : t - \Delta < s < t \text{ and } |v| > P \text{ and } |v - \widehat{V}(t)| > P \\ &\qquad \text{and } |X(s, t, x, v) - \widehat{X}(s)| > R\big\}. \end{aligned}$$

We will use the invertibility of the characteristics as described in Glassey (1996), so that when we set

$$\begin{aligned} y &= X(s, t, x, v), \\ w &= V(s, t, x, v) \end{aligned}$$

we can invert using

$$\begin{aligned} x &= X(t, s, y, w), \\ v &= V(t, s, y, w). \end{aligned}$$

In particular, notice $w = V(s, t, X(t, s, y, w), V(t, s, y, w))$ and $\frac{\partial(y,w)}{\partial(x,v)} = 1$.

To handle the integral over G, we must first deal with some preliminary inequalities:

1. First notice that $|V(s, t, x, v) - v| \le \int_s^t |E(\tau, X(\tau)) + A(\tau, X(\tau))| d\tau$.
   Thus, for $s \in [t - \Delta, t]$, we have

$$|V(s, t, x, v) - v| \le \Delta C^{(2)} Q^{\frac{4}{3}}(t) = \frac{1}{4} P.$$

2. For $|v| < P$,

$$|V(s, t, x, v)| \le |v| + \frac{1}{4} P < 2P.$$

3. For $|v - \widehat{V}(t)| < P$,

$$\begin{aligned} |V(s, t, x, v) - \widehat{V}(s)| &\le |v - \widehat{V}(t)| + |\widehat{V}(t) - \widehat{V}(s)| + |V(s, t, x, v) - v| \\ &\le P + \frac{1}{4} P + \frac{1}{4} P < 2P. \end{aligned}$$

4. For $|v| > P$ and $\tau \in [t-\Delta, t]$,

$$|V(\tau, t, x, v)| \geq |v| - \frac{1}{4}P > \frac{3}{4}P \geq \frac{3}{4} \cdot 2W > W.$$

Now, let

$$\chi_G(s, x, v) := \begin{cases} 1 & (s, x, v) \in G \\ 0 & \text{else.} \end{cases}$$

Then, we have

$$\begin{aligned} I_G &= \iint_G \int \frac{|g(s, X(s,t,x,v), V(s,t,x,v))|}{|X(s,t,x,v) - \widehat{X}(s)|^2} dv\, dx\, ds \\ &= \int_{t-\Delta}^t \iint \frac{\chi_G(s,x,v)|g(s, X(s,t,x,v), V(s,t,x,v))|}{|X(s,t,x,v) - \widehat{X}(s)|^2} dv\, dx\, ds \\ &= \int_{t-\Delta}^t \iint \frac{\chi_G(s, X(t,s,y,w), V(t,s,y,w))|g(s,y,w)|}{|y - \widehat{X}(s)|^2} dw\, dy\, ds. \end{aligned}$$

If $\chi_G \neq 0$, then $|V(t,s,y,w)| < P$ or $|V(t,s,y,w) - \widehat{V}(t)| < P$. Then, by Preliminary Inequalities 2 and 3, we have either $|w| < 2P$ or $|w - \widehat{V}(s)| < 2P$. Set

$$\tilde{\rho}(s, y) := \int |g(s,y,w)| \chi_G(s, X(t,s,y,w), V(t,s,y,w)) dw.$$

Then, $\|\tilde{\rho}(s)\|_{L^\infty} \leq CP^3$. Also, using (13), we know

$$\tilde{\rho}(s,y) \leq \int |g(s,y,w)| dw \leq C(k^{\frac{3}{5}}(s,y) + k^{\frac{1}{2}}(s,y)).$$

Thus, we employ the method of (16) and (17) to find

$$\int \frac{\tilde{\rho}(s,y)}{|y - \widehat{X}(s)|^2} dy \leq CP^{\frac{4}{3}}$$

and, finally, we have

$$I_G = \int_{t-\Delta}^t \int \frac{\tilde{\rho}(s,y)}{|y - \widehat{X}(s)|^2} dy\, ds \leq C\Delta P^{\frac{4}{3}}. \tag{22}$$

Estimating I_B, we have

$$\begin{aligned} I_B &= \iint_B \int \frac{|g(s, X(s,t,x,v), V(s,t,x,v))|}{|X(s,t,x,v) - \widehat{X}(s)|^2} dv\, dx\, ds \\ &\leq \int_{t-\Delta}^t \int_{|y - \widehat{X}(s)| < R} \int \frac{|g(s,y,w)|}{|y - \widehat{X}(s)|^2} dw\, dy\, ds \end{aligned}$$

$$\leq (\|F\|_{L^\infty} + \|f_0\|_{L^\infty}) Q^3(t) \int_{t-\Delta}^{t} \int_{|y-\widehat{X}(s)|<R} |y - \widehat{X}(s)|^{-2} dw \, ds$$
$$\leq C Q^3(t) \int_{t-\Delta}^{t} \int_{0}^{R} dr \, ds,$$

and thus

$$I_B \leq C \Delta Q^3(t) R. \tag{23}$$

Finally, to estimate I_U, we use Section 3 (specifically, line (15)) of Schaeffer (1991) to find

$$\int_{t-\Delta}^{t} |X(s,t,x,v) - \widehat{X}(s)|^{-2} \chi_U(s,x,v) ds \leq \frac{C}{RP}, \tag{24}$$

for (x, v) as in U. The proof of this result is quite long, so we shall include a sketch rather than the entire proof. First, let

$$Z(s) = X(s,t,x,v) - \widehat{X}(s).$$

Then, choose $s_0 \in [t-\Delta, t]$ such that

$$|Z(s_0)| \leq |Z(s)|$$

for all $s \in [t-\Delta, t]$. It is shown in Schaeffer (1991) that

$$|Z(s)| \geq \frac{1}{4} P |s - s_0|. \tag{25}$$

Now, define

$$\Sigma(r) := \begin{cases} \dfrac{1}{R^2} & 0 \leq r \leq R^2 \\ \dfrac{1}{r} & r \geq R^2. \end{cases}$$

Notice that Σ is non-negative, non-increasing and

$$|Z(s)|^{-2} \chi_U(s,x,v) \leq \Sigma(|Z(s)|^2).$$

Using these properties of Σ with (25), we find

$$\int_{t-\Delta}^{t} |Z(s)|^{-2} \chi_U(s,x,v) ds \leq \int_{t-\Delta}^{t} \Sigma(|Z(s)|^2) ds \leq \int_{t-\Delta}^{t} \Sigma\left(\left(\frac{1}{4} P |s - s_0|\right)^2\right) ds$$
$$\leq \int \Sigma\left(\frac{P^2}{16}\tau^2\right) d\tau = \frac{16}{PR}.$$

This shows (24).

Now, using (24) with (3), (9), (19), and preliminary inequality 4, we have

$$
\begin{aligned}
I_U &= \iint_U \int \frac{|g(s, X(s,t,x,v), V(s,t,x,v))|}{|X(s,t,x,v) - \widehat{X}(s)|^2} dv\, dx\, ds \\
&= \iint_U \int \frac{|F(V(s,t,x,v)) - f(s, X(s,t,x,v), V(s,t,x,v))|}{|X(s,t,x,v) - \widehat{X}(s)|^2} dv\, dx\, ds \\
&= \iint_U \int \frac{|-f(t,x,v)|}{|X(s,t,x,v) - \widehat{X}(s)|^2} dv\, dx\, ds \\
&= \int_{t-\Delta}^{t} \int_{|v|>P \cap |v-\widehat{V}(t)|>P} \int_{|X(s,t,x,v)-\widehat{X}(s)|>R} \frac{f(t,x,v)}{|X(s,t,x,v) - \widehat{X}(s)|^2} dx\, dv\, ds \\
&= \int_{|v|>P} \int f(t,x,v) \left(\int_{t-\Delta}^{t} |X(s,t,x,v) - \widehat{X}(s)|^{-2} \chi_U(s,x,v) ds \right) dx\, dv \\
&\leq \frac{C}{RP} \iint_{|v|>P} f(t,x,v) dv\, dx \\
&\leq \frac{C}{RP^3} \iint_{|v|>P} |v|^2 f(t,x,v) dv\, dx
\end{aligned}
$$

and so

$$
I_U \leq \frac{C}{RP^3}. \tag{26}
$$

Finally, collecting the estimates (22), (23), and (26), we find,

$$
\frac{1}{\Delta} \int_{t-\Delta}^{t} |E(s, \widehat{X}(s))| ds \leq C \left(P^{\frac{4}{3}} + RQ^3(t) + \frac{1}{\Delta RP^3} \right).
$$

We take $R = Q^{-\frac{32}{15}}(t)$ and then

$$
\frac{1}{\Delta} \int_{t-\Delta}^{t} |E(s, \widehat{X}(s))| ds \leq CQ^{\frac{13}{15}}(t).
$$

Using (III), we have

$$
\begin{aligned}
&\frac{1}{\Delta} \int_{t-\Delta}^{t} \left(|E(s, \widehat{X}(s)) + A(s, \widehat{X}(s))| \right) ds \\
&\quad \leq CQ^{\frac{13}{15}}(t) + \frac{1}{\Delta} \int_{t-\Delta}^{t} C^{(0)} R^{-2}(\widehat{X}(s)) ds \leq CQ^{\frac{13}{15}}(t) + \frac{1}{\Delta} \int_{t-\Delta}^{t} C^{(0)} ds \\
&\quad \leq CQ^{\frac{13}{15}}(t) + \left((C^{(0)})^{\frac{15}{13}} \right)^{\frac{13}{15}} \leq CQ^{\frac{13}{15}}(t) + Q^{\frac{13}{15}}(t) \leq CQ^{\frac{13}{15}}(t).
\end{aligned} \tag{27}
$$

Finally, we use the argument in Section 4.5 of Glassey (1996) to bound the velocity support, since the power of $Q(t)$ is less than one. This work will be explored in Appendix B. Thus, for all $t \in [0, T]$,

$$
Q(t) \leq C, \tag{28}
$$

and this implies bounds on $Q_f(t)$ and $Q_g(t)$ for all $t \in [0, T]$. Furthermore, if $V(s, t, x, v)$ satisfies $f_0(X(0, t, x, v), V(0, t, x, v)) \neq 0$, then

$$|V(s, t, x, v)| \leq C$$

for any $s \in [0, T]$, including $V(t, t, x, v) = v$.

Notice then, the bound on the velocity support implies a priori bounds on $\|\rho(t)\|_{L^\infty}$, $\|E(t)\|_{L^\infty}$, $\|\nabla_x E(t)\|_{L^\infty}$, and $\|\nabla_{x,v} f(t)\|_{L^\infty}$ for all $t \in [0, T]$.

Now that $\|\rho(t)\|_\infty \leq C$ and $\|\nabla\rho(t)\|_\infty \leq C$ for all $t \in [0, T]$, and $Q_f(T)$ is finite, we apply Lemma 4 since $p > 3$, and find $\|\rho(t)\|_p \leq C$ for all $t \in [0, T]$, and the proof of Theorem 1 is complete.

3. Section 3

To conclude the article, this section contains the proofs of Lemmas 2 through 4.

Proof of Lemma 2. Let $q, T > 0$ be given with $\|\rho(t)\|_\infty \leq C$ for all $t \in [0, T]$. Let $b \in [0, \frac{5}{18})$ be given with $b \leq \frac{2}{q}$. Consider $|x| \geq 1$, define

$$\eta := (\|\rho(t)\|_q R^{-q}(x))^b$$

and divide the field into the pieces

$$|E(t, x)| \leq \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

where

$$\mathrm{I} := \int_{|x-y|<\eta} |\rho(t, y)||x-y|^{-2} dy,$$
$$\mathrm{II} := \int_{\eta<|x-y|<\frac{1}{2}|x|} |\rho(t, y)||x-y|^{-2} dy,$$
$$\mathrm{III} := \int_{|x-y|>\frac{1}{2}|x|} |\rho(t, y)||x-y|^{-2} dy.$$

Then, the first estimate satisfies

$$\mathrm{I} \leq C\|\rho(t)\|_\infty \eta. \tag{29}$$

The second estimate satisfies, for any $m \in [0, \frac{1}{3})$,

$$\begin{aligned}
\mathrm{II} &\leq C(\|\rho(t)\|_q R^{-q}(x))^m \int_{\eta<|x-y|<\frac{1}{2}|x|} |\rho(t, y)|^{1-m}|x-y|^{-2} dy \\
&\leq C(\|\rho(t)\|_q R^{-q}(x))^m \int_{\eta<|x-y|<\frac{1}{2}|x|} \Big(k^{\frac{3(1-m)}{5}}(t, y)|x-y|^{-2} + k^{\frac{1-m}{2}}(t, y)|x-y|^{-2}\Big) dy \\
&\leq C(\|\rho(t)\|_q R^{-q}(x))^m \Bigg[\Big(\int_{\eta<|x-y|<\frac{1}{2}|x|} |x-y|^{-\frac{4}{1+m}} dy\Big)^{\frac{1+m}{2}} \\
&\qquad + \Big(\int_{\eta<|x-y|<\frac{1}{2}|x|} |x-y|^{-\frac{10}{3m+2}} dy\Big)^{\frac{3m+2}{5}}\Bigg]
\end{aligned}$$

$$\leq C(\|\rho(t)\|_q R^{-q}(x))^m \left[\eta^{-2+3(\frac{1+m}{2})} + \eta^{-2+3(\frac{3m+2}{5})}\right]$$

$$\leq C(\|\rho(t)\|_q R^{-q}(x))^m \begin{cases} \eta^{\frac{3m-1}{2}}, & \eta \geq 1 \\ \eta^{\frac{9m-4}{5}}, & \eta \leq 1 \end{cases}$$

Then, for $\eta \geq 1$, we choose $m = \frac{3b}{2+3b}$, and for $\eta \leq 1$, we choose $m = \frac{9b}{5+9b}$. To guarantee convergence of the above integrals, we must have $m < \frac{1}{3}$, and thus, $b < \frac{5}{18}$. Thus, we find

$$\mathrm{II} \leq C(\|\rho(t)\|_q R^{-q}(x))^b. \tag{30}$$

Finally, for $b \leq \frac{2}{q}$,

$$\begin{aligned}
\mathrm{III} &\leq \|\rho(t)\|_q^b \left(\frac{1}{2}|x|\right)^{-qb} \int_{|x-y|>\frac{1}{2}|x|} |\rho(t,y)|^{1-b}|x-y|^{bq-2}\ R^{-bq}(y)dy \\
&\leq C(\|\rho(t)\|_q R^{-q}(x))^b \int_{|x-y|>\frac{1}{2}|x|} |\rho(t,y)|^{1-b}\left(\frac{1}{4}R(y)\right)^{bq-2} R^{-bq}(y)dy \\
&\leq C(\|\rho(t)\|_q R^{-q}(x))^b \int_{|x-y|>\frac{1}{2}|x|} (k^{\frac{1-b}{2}}(t,y) + k^{\frac{3}{5}(1-b)}(t,y))R^{-2}(y)dy \\
&\leq C(\|\rho(t)\|_q R^{-q}(x))^b \left[\left(\int R^{-\frac{4}{1+b}}(y)dy\right)^{\frac{1+b}{2}} + \left(\int R^{-\frac{10}{3b+2}}(y)dy\right)^{\frac{3b+2}{5}}\right] \\
&\leq C(\|\rho(t)\|_q R^{-q}(x))^b
\end{aligned}$$

for $-\frac{4}{1+b} < -3$, which is satisfied since $b < \frac{5}{18}$.

Combining the estimates for I, II, and III, the lemma follows.

Proof of Lemma 3. Let $q, T > 0$ be given with $\|\rho(t)\|_\infty \leq C$ and $\|\nabla\rho(t)\|_\infty \leq C$ for all $t \in [0, T]$. Let $a \in [0, 1)$ be given with $a \leq \frac{3}{q}$. Consider $|x| \geq 1$ and define

$$\eta := (\|\rho(t)\|_q R^{-q}(x))^a.$$

For any $i, k = 1, 2, 3$, we have

$$\begin{aligned}
|\partial_{x_i} E_k(t,x)| &= \left|\int_{|x-y|<\eta} \partial_{y_i}\rho(t,y)\frac{(x-y)_k}{|x-y|^3}dy\right| + \left|\int_{|x-y|=\eta} \rho(t,y)\frac{(x-y)_k}{|x-y|^3}\frac{(x-y)_i}{|x-y|}dS_y\right| \\
&\quad + \left|\int_{|x-y|>\eta} \rho(t,y)\partial_{y_i}\left(\frac{(x-y)_k}{|x-y|^3}\right)dy\right| \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Then,

$$\mathrm{I} \leq \int_{|x-y|<\eta} \|\nabla_x\rho(t)\|_\infty |x-y|^{-2}dy \leq C\eta.$$

We estimate II for the large $|x|$ and small $|x|$ cases. For $|x| > 2\eta$,

$$\begin{aligned}
\mathrm{II} &\leq \int_{|x-y|=\eta} (\|\rho(t)\|_q R^{-q}(y))^a |\rho(t,y)|^{1-a} |x-y|^{-2} dS_y \\
&\leq \|\rho(t)\|_q^a \|\rho(t)\|_{L^\infty}^{1-a} \eta^{-2} \int_{|x-y|=\eta} R^{-aq}(|x|-\eta) dS_y \\
&\leq C\|\rho(t)\|_q^a \eta^{-2} R^{-aq}\left(\frac{1}{2}|x|\right)\eta^2 \leq C(\|\rho(t)\|_q R^{-q}(x))^a.
\end{aligned}$$

For $|x| < 2\eta$,

$$\mathrm{II} \leq \|\rho(t)\|_\infty \eta^{-2} \int_{|x-y|=\eta} dS_y \leq C \leq C(2\eta|x|^{-1}) \leq C\eta.$$

Then,

$$\begin{aligned}
\mathrm{III} \leq\ & C\int_{\eta<|x-y|<\frac{1}{2}|x|} |\rho(t,y)||x-y|^{-3} dy \\
& + C\int_{|x-y|>\frac{1}{2}|x|} |\rho(t,y)||x-y|^{-3} dy =: A + B.
\end{aligned}$$

Estimating A, we have for $n \in [0, 1)$,

$$\begin{aligned}
A &\leq \|\rho(t)\|_q^n \int_{\eta<|x-y|<\frac{1}{2}|x|} |\rho(t,y)|^{1-n} R^{-nq}(y)|x-y|^{-3} dy \\
&\leq C\|\rho(t)\|_q^n R^{-nq}\left(\frac{1}{2}|x|\right) \int_{\eta<|x-y|<\frac{1}{2}|x|} \left(k^{\frac{1-n}{2}}(t,y) + k^{\frac{3(1-n)}{5}}(t,y)\right)|x-y|^{-3} dy \\
&\leq C(\|\rho(t)\|_q R^{-q}(x))^n \Bigg[\left(\int_{\eta<|x-y|<\frac{1}{2}|x|} |x-y|^{-\frac{6}{1+n}}\, dy\right)^{\frac{1+n}{2}} \\
&\qquad\qquad + \left(\int_{\eta<|x-y|<\frac{1}{2}|x|} |x-y|^{-\frac{15}{3n+2}}\right)^{\frac{3n+2}{5}}\Bigg] \\
&\leq C(\|\rho(t)\|_q R^{-q}(x))^n \left[\left(\int_\eta^\infty r^{-\frac{6}{1+n}+2} dr\right)^{\frac{1+n}{2}} + \left(\int_\eta^\infty r^{-\frac{15}{3n+2}+2} dr\right)^{\frac{3n+2}{5}}\right] \\
&\leq C(\|\rho(t)\|_q R^{-q}(x))^n \left(\eta^{\frac{3}{2}(1+n)-3} + \eta^{\frac{3}{5}(3n+2)-3}\right) \\
&\leq C(\|\rho(t)\|_q R^{-q}(x))^n \left(\eta^{-\frac{3}{2}(1-n)} + \eta^{-\frac{9}{5}(1-n)}\right) \\
&\leq C(\|\rho(t)\|_q R^{-q}(x))^n \begin{cases} \eta^{-\frac{9}{5}(1-n)}, & \eta \leq 1 \\ \eta^{-\frac{3}{2}(1-n)}, & \eta \geq 1. \end{cases}
\end{aligned}$$

For $\eta \leq 1$, we may choose $n = \frac{14a}{9a+5}$, and for $\eta \geq 1$, we may choose $n = \frac{5a}{3a+2}$. Thus,

$$A \leq C(\|\rho(t)\|_q R^{-q}(x))^a.$$

Finally, we estimate B and find

$$
\begin{aligned}
B &\leq \int_{|x-y|>\frac{1}{2}|x|} |\rho(t,y)||x-y|^{-3}dy \\
&\leq \|\rho(t)\|_q^a \left(\frac{1}{2}|x|\right)^{-aq} \int_{|x-y|>\frac{1}{2}|x|} |\rho(t,y)|^{1-a}|x-y|^{aq-3}R^{-aq}(y)dy \\
&\leq C(\|\rho(t)\|_q R^{-q}(x))^a \int_{|x-y|>\frac{1}{2}|x|} \left(k^{\frac{1-a}{2}}(t,y) + k^{\frac{3}{5}(1-a)}(t,y)\right) R^{-3}(y)dy \\
&\leq C(\|\rho(t)\|_q R^{-q}(x))^a \left[\left(\int R^{-\frac{6}{1+a}}(y)dy\right)^{\frac{1+a}{2}} + \left(\int R^{-\frac{15}{3a+2}}(y)dy\right)^{\frac{3a+2}{5}}\right] \\
&\leq C(\|\rho(t)\|_q R^{-q}(x))^a
\end{aligned}
$$

for $-\frac{6}{1+a} < -3$, which is satisfied since $a < 1$.

Combining the estimates for I, II, A, and B, the lemma follows.

Proof of Lemma 4. Let $T > 0$ and $q \in [0, \frac{54}{13})$ be given with $\|\rho(t)\|_\infty \leq C$ and $\|\nabla\rho(t)\|_\infty \leq C$ for all $t \in [0, T]$ and $Q_f(T) < \infty$. Let $t \in [0, T]$ be given. For any $D > 0$, taking $|x| \leq D$, we find

$$
|\rho(t,x)| \leq \|\rho(t)\|_\infty \leq \|\rho(t)\|_\infty \frac{R^q(D)}{R^q(x)} \leq CR^{-q}(x). \tag{31}
$$

Now, define $C^{(3)} := \max\{1, 8TQ_g(T)\}$ and let $|x| \geq C^{(3)}$. Define for every $t \in [0, T]$ and $x \in \mathbb{R}^3$,

$$
\mathscr{E}(t,x) = E(t,x) + A(t,x).
$$

From the Vlasov equation,

$$
\frac{\partial}{\partial s}(g(s, X(s), V(s))) = -\mathscr{E}(s, X(s)) \cdot \nabla_v F(V(s)),
$$

and thus

$$
g(t,x,v) = g(0, X(0), V(0)) - \int_0^t \mathscr{E}(s, X(s)) \cdot \nabla_v F(V(s))ds. \tag{32}
$$

Thus, to estimate ρ, we must consider $\int \mathscr{E}(s, X(s)) \cdot \nabla_v F(V(s))dv$. Assume f is nonzero along $(X(s), V(s))$. Then,

$$
\begin{aligned}
\left|\int \mathscr{E}(s, X(s)) \cdot \nabla_v F(V(s))dv\right| \leq & \left|\int \mathscr{E}(s, X(s)) \cdot \left(\nabla_v F(V(s)) - \nabla_v F\left(v + \int_s^t \mathscr{E}(\tau, x)d\tau\right)\right)dv\right| \\
& + \left|\int (\mathscr{E}(s, X(s)) - \mathscr{E}(s, x + (s-t)v)) \right. \\
& \left. \cdot \nabla_v F(v + \int_s^t \mathscr{E}(\tau, x)d\tau)dv\right|
\end{aligned}
$$

$$
\begin{aligned}
&+ \left| \int \nabla_v \cdot (F(v)\ \mathscr{E}(s, x + (s-t)v))dv \right| \\
&+ \left| \int F(v) \nabla_v \cdot (\mathscr{E}(s, x + (s-t)v))dv \right| \\
=:\ & \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}
$$

Using Lemmas 2 and 3, as well as (III), we find $a \in [0, 1)$ and $b \in [0, \frac{5}{18})$ with $a \leq \frac{3}{q}$, $b \leq \frac{2}{q}$, and $a + b = 1$ such that

$$|\mathscr{E}(t, x)| \leq C(\|\rho(t)\|_q R^{-q}(x))^b$$

and

$$|\nabla \mathscr{E}(t, x)| \leq C(\|\rho(t)\|_q R^{-q}(x))^a.$$

By the Mean Value Theorem, for $\tau \in [s, t]$ and $i = 1, 2, 3$, there exist ξ_1^i on the line segment between $X(\tau)$ and x such that

$$\mathscr{E}_i(\tau, X(\tau)) - \mathscr{E}_i(\tau, x) = \nabla_x \mathscr{E}_i(\tau, \xi_1^i) \cdot (X(\tau) - x).$$

Hence,

$$
\begin{aligned}
\mathrm{I} &\leq \int_{|v| \leq Q_g(T)} |\mathscr{E}(s, X(s))| \|\nabla^2 F\|_\infty \left| V(s) - \left(v + \int_s^t \mathscr{E}(\tau, x) d\tau \right) \right| dv \\
&\leq C \int_{|v| \leq Q_g(T)} (\|\rho(s)\|_q R^{-q}(X(s)))^b \left| \int_s^t (\mathscr{E}(\tau, X(\tau)) - \mathscr{E}(\tau, x)) d\tau \right| dv \\
&\leq C \int_{|v| \leq Q_g(T)} (\|\rho(s)\|_q R^{-q}(X(s)))^b \int_s^t \sup_i |\nabla \mathscr{E}(\tau, \xi_1^i)| |X(\tau) - x| d\tau\, dv \\
&\leq C \int_{|v| \leq Q_g(T)} (\|\rho(s)\|_q R^{-q}(X(s)))^b T Q_g(T) \int_s^t \sup_i (\|\rho(\tau)\|_q R^{-q}(\xi_1^i))^a d\tau\, dv.
\end{aligned}
$$

Since we know, by (11), for any $i = 1, 2, 3$

$$|\xi_1^i| \geq |X(\tau)| - |X(\tau) - x| \geq \frac{1}{2}|x| - T Q_g(T) \geq \frac{1}{4}|x|,$$

we find

$$
\begin{aligned}
\mathrm{I} &\leq C\left(\|\rho(s)\|_q R^{-q}\left(\frac{1}{2}|x|\right)\right)^b R^{-aq}\left(\frac{1}{4}|x|\right) \int_s^t \|\rho(\tau)\|_q^a d\tau \\
&\leq C R^{-q(a+b)}(x) \left(\|\rho(s)\|_q^b \int_s^t \|\rho(\tau)\|_q^a d\tau \right).
\end{aligned}
$$

Similarly, using the above lemmas and the Mean Value Theorem, for any $i = 1, 2, 3$ there exist ξ_2^i between $X(\tau)$ and $x + (s-t)v$ such that

$$\begin{aligned}
\mathrm{II} &\le C \sum_{i=1}^{3} \int_{|v| \le Q_g(T)} \sup_i |\nabla \mathscr{E}(s, \xi_2^i)| |X(s) - (x + (s-t)v)| \|\nabla F\|_\infty dv \\
&\le C \sum_{i=1}^{3} \int_{|v| \le Q_g(T)} \sup_i (\|\rho(s)\|_q R^{-q}(\xi_2^i))^a \left| \int_s^t \int_\tau^t \mathscr{E}(\iota, X(\iota)) d\iota \, d\tau \right| dv.
\end{aligned}$$

Since we know, using (11), for any $i = 1, 2, 3$

$$\begin{aligned}
|\xi_2^i| &\ge |X(s)| - |X(s) - (x + (s-t)v)| \ge \frac{1}{2}|x| - \left| \int_s^t (V(\tau) - v) d\tau \right| \\
&\ge \frac{1}{2}|x| - 2TQ_g(T) \ge \frac{1}{4}|x|,
\end{aligned}$$

it follows that

$$\begin{aligned}
\mathrm{II} &\le C \int_{|v| \le Q_g(T)} \left(\|\rho(s)\|_q R^{-q}\left(\frac{1}{4}|x|\right)\right)^a \int_s^t (\|\rho(s)\|_q R^{-q}(X(\tau)))^b d\tau \, dv \\
&\le C R^{-q(a+b)}(x) \Big(\|\rho(s)\|_q^a \int_s^t \|\rho(\tau)\|_q^b \, d\tau \Big).
\end{aligned}$$

By the Divergence Theorem,

$$\mathrm{III} = 0,$$

and finally,

$$\begin{aligned}
\mathrm{IV} &\le \int_{|v| \le Q_g(T)} \|F\|_\infty |\nabla_x \cdot \mathscr{E}(s, x + (s-t)v)| \, |s-t| dv \\
&\le C \int_{|v| \le Q_g(T)} |\rho(s, x + (s-t)v)| dv.
\end{aligned}$$

Collecting the estimates for I–IV, we have

$$\begin{aligned}
&\left| \int \mathscr{E}(s, X(s)) \cdot \nabla_v F(V(s)) dv \right| \\
&\quad \le C \Big(R^{-q(a+b)}(x) \Big(\|\rho(s)\|_q^b \int_s^t \|\rho(\tau)\|_q^a d\tau \Big) + R^{-q(a+b)}(x) \Big(\|\rho(s)\|_q^a \int_s^t \|\rho(\tau)\|_q^b d\tau \Big) \\
&\qquad + \int_{|v| \le Q_g(T)} |\rho(s, x + (s-t)v)| dv \Big).
\end{aligned} \tag{33}$$

Since $a + b = 1$, we proceed from (32) and using (33), (IV), and (28), we find

$$\begin{aligned}
|\rho(t,x)| &= \left| \int g(t,x,v) dv \right| \\
&\le \int_{|v| \le Q_g(T)} |g(0, X(0), V(0))| dv + \left| \int_0^t \int_{|v| \le Q_g(T)} \mathscr{E}(s, X(s)) \cdot \nabla_v F(V(s)) dv \, ds \right|
\end{aligned}$$

$$\leq CR^{-q}(x) + C\int_0^t \left(R^{-q(a+b)}(x)\Big(\|\rho(s)\|_q^b \int_s^t \|\rho(\tau)\|_q^a d\tau\Big) + R^{-q(a+b)}(x)\Big(\|\rho(s)\|_q^a \int_s^t \|\rho(\tau)\|_q^b d\tau\Big) + \int_{|v|\leq Q_g(T)} |\rho(s, x+(s-t)v)|dv\right)ds$$

$$\leq CR^{-q}(x)\Big(1 + \int_0^t \|\rho(s)\|_q^b \int_s^t \|\rho(\tau)\|_q^a d\tau\, ds + \int_0^t \|\rho(s)\|_q^a \int_s^t \|\rho(\tau)\|_q^b d\tau\, ds\Big) + C\int_{|v|\leq Q_g(T)} \int_0^t |\rho(s, x+(s-t)v)|ds\, dv$$

$$\leq CR^{-q}(x)\Big(1 + \Big(\int_0^t \|\rho(s)\|_q^b ds\Big)\Big(\int_0^t \|\rho(\tau)\|_q^a d\tau\Big) + \Big(\int_0^t \|\rho(s)\|_q^a ds\Big)\Big(\int_0^t \|\rho(\tau)\|_q^b d\tau\Big)\Big) + C\int_{|v|\leq Q_g(T)} \int_0^t |\rho(s, x+(s-t)v)|ds\, dv.$$

Then, applying Hölder's inequality twice,

$$|\rho(t,x)| \leq CR^{-q}(x)\Big(1 + 2T^2\Big(\int_0^t \|\rho(s)\|_q ds\Big)^{a+b} + R^q(x)\int_{|v|\leq Q_g(T)} \int_0^t \|\rho(s)\|_q R^{-q}(x+(s-t)v)ds\, dv\Big)$$

$$\leq CR^{-q}(x)\Big(1 + \int_0^t \|\rho(s)\|_q ds + R^q(x)\; R^{-q}\Big(\frac{7}{8}|x|\Big)Q_g^3(T)\; \int_0^t \|\rho(s)\|_q ds\Big)$$

$$\leq CR^{-q}(x)\Big(1 + \int_0^t \|\rho(s)\|_q ds\Big).$$

Combining this with (31) (with $D = C^{(3)}$) and multiplying by $R^q(x)$, we have for all x,

$$R^q(x)|\rho(t,x)| \leq C\Big(1 + \int_0^t \|\rho(s)\|_q ds\Big).$$

Since the right side of the inequality is independent of x, we take the supremum over all x to find,

$$\|\rho(t)\|_q \leq C\Big(1 + \int_0^t \|\rho(s)\|_q ds\Big)$$

and by Gronwall's Inequality,

$$\|\rho(t)\|_q \leq C.$$

Thus, $\|\rho(t)\|_q$ is bounded for all $t \in [0, T]$. Notice, too, that the choice of $a + b = 1$ forces $q < \frac{54}{13}$, and the proof of the lemma is complete.

Appendix A

In this appendix, we will explore the argument used in Section 4.2.6 of Glassey (1996) to bound derivatives of f and E.

Let D be any x derivative. Then, using the Vlasov equation,

$$\partial_t(Df) + v \cdot \nabla_x(Df) - E \cdot \nabla_v(Df) = DE \cdot \nabla_v f$$

so that

$$\frac{d}{ds} Df(s, X(s), V(s)) = DE \cdot \nabla_v f(s, X(s), V(s)).$$

Hence,

$$|Df(t, x, v)| \leq |Df(0, X(0), V(0))| + \int_0^t |DE \cdot \nabla_v f(s, X(s), V(s))| ds.$$

Define

$$|f(s)|_1 = \sup_{x,v} |\partial_x f(s, x, v)| + \sup_{x,v} |\partial_v f(t, x, v)|$$

and

$$|E(s)|_1 = \sup_x |\partial_x E(s, x)|.$$

Then,

$$|Df(t, x, v)| \leq C + \int_0^t |E(s)|_1 |f(s)|_1 ds.$$

We see that $\partial_v f$ satisfies an inequality of similar type because

$$\partial_t(\partial_v f) + v \cdot \nabla_x(\partial_v f) - E \cdot \nabla_v(\partial_v f) = -\partial_v v \cdot \nabla_x f.$$

It follows that

$$|f(t)|_1 \leq C + \int_0^t (1 + |E(s)|_1)|f(s)|_1 ds.$$

However, from (18), we can conclude (with $d = \|\nabla_x \rho\|_\infty$) that

$$|E(s)|_1 \leq C(1 + \ln^* |f(s)|_1)$$

where

$$\ln^* s = \begin{cases} s & 0 \leq s \leq 1 \\ 1 + \ln s & s > 1. \end{cases}$$

Therefore, for $t \leq T$,

$$|f(t)|_1 \leq C\Big(1 + \int_0^t (1 + \ln^* |f(s)|_1)|f(s)|_1 ds.\Big).$$

It now follows by an application of Gronwall's Iequality that

$$|f(t)|_1 \le C$$

and

$$|E(t)|_1 \le C$$

on $t \le T$. This concludes the argument to bound field derivatives and derivatives of the density, and thus ends Appendix A.

Appendix B

The bound on $Q(t)$ is obtained as follows. From (8) and (27), we have

$$|\widehat{V}(t)| \le |\widehat{V}(t-\Delta)| + \int_{t-\Delta}^{t} |E(s, \widehat{X}(s))|ds \le Q(t-\Delta) + C\Delta Q^{\frac{13}{15}}(t).$$

Since the above constant is independent of the particular characteristic, we find

$$Q(t) \le Q(t-\Delta) + C\Delta Q^{\frac{13}{15}}(t)$$

for

$$\Delta = \min\left\{t, \frac{1}{4C^{(2)}}Q^{-\frac{4}{3}}(t)\cdot Q^{\frac{13}{20}}(t)\right\} = \min\left\{t, \frac{1}{4C^{(2)}}Q^{-\frac{41}{60}}(t)\right\}.$$

Since Q is nondecreasing, there exists T_1 such that

$$\Delta = \begin{cases} t & t \le T_1 \\ \frac{1}{4C^{(2)}}Q^{-\frac{41}{60}}(t) & t \ge T_1. \end{cases}$$

Take t_0 in the interval of existence. Without loss of generality, $t_0 > T_1$. Let

$$t_1 = t_0 - \frac{1}{4C^{(2)}}Q^{-\frac{41}{60}}(t_0),$$

$$t_{i+1} = t_i - \frac{1}{4C^{(2)}}Q^{-\frac{41}{60}}(t_i) \quad (i = 1, 2, \dots)$$

as long as $t_i > T_1$. Then

$$t_i - t_{i+1} = \frac{1}{4C^{(2)}}Q^{-\frac{41}{60}}(t_i) \ge \frac{1}{4C^{(2)}}Q^{-\frac{41}{60}}(t_0)$$

which is a uniform lower bound on the length of each subinterval. So, there is a first i, say $i = k$, such that $t_k \le T_1$. Thus, $t_k \ge 0$ and therefore

$$(t_0 - t_1) + (t_1 - t_2) + \cdots + (t_{k-1} - t_k) \ge k \cdot \frac{1}{4C^{(2)}}Q^{-\frac{41}{60}}(t_0),$$

which implies that

$$Q^{-\frac{41}{60}}(t_0)\cdot k \le 4C^{(2)}t_0.$$

Now we have

$$Q(t_0) = Q(t_k) + \sum_{i=0}^{k-1}[Q(t_i) - Q(t_{i+1})] \leq Q(t_k) + C\sum_{i=0}^{k-1} \Delta \cdot Q^{\frac{13}{15}}(t_i)$$
$$\leq Q(T_1) + C\sum_{i=0}^{k-1} \frac{1}{4C^{(2)}} Q^{-\frac{41}{60}}(t_0) \cdot Q^{\frac{13}{15}}(t_i)$$
$$\leq Q(T_1) + C \cdot kQ^{-\frac{41}{60}}(t_0) \cdot Q^{\frac{13}{15}}(t_0) \leq Ct_0 Q^{\frac{13}{15}}(t_0).$$

Therefore, $Q(t_0)$ is bounded, and the proof is complete. This ends Appendix B.

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