Completely Bounded Maps and Operator Algebras
Chapter 1: Introduction
Solutions to Exercises

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1.1 Let $(T_{i,j})$ be in $M_n(B(\mathcal{H}))$. Verify that the linear transformation it determines on $\mathcal{H}^{(n)}$ is bounded and that, in fact, $\|(T_{i,j})\| \leq \left(\sum_{i,j=1}^n \|T_{i,j}\|^2\right)^{1/2}$.

Solution: Let $\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} \in \mathcal{H}^{(n)}$ be of unit length. We compute

$$\begin{aligned}
\left\|(T_{i,j})\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}\right\|^2 &= \left\|\begin{bmatrix} \sum_{j=1}^n T_{1,j}h_j \\ \vdots \\ \sum_{j=1}^n T_{n,j}h_j \end{bmatrix}\right\|^2 = \sum_{i=1}^n \left\|\sum_{j=1}^n T_{i,j}h_j\right\|^2 && (1)\\
&\leq \sum_{i=1}^n \left(\sum_{j=1}^n \|T_{i,j}h_j\|\right)^2 \quad \text{triangle inequality} && (2)\\
&\leq \sum_{i=1}^n \left(\sum_{j=1}^n \|T_{i,j}\|\,\|h_j\|\right)^2 && (3)\\
&\leq \sum_{i=1}^n \left[\left(\sum_{j=1}^n \|T_{i,j}\|^2\right)\left(\sum_{j=1}^n \|h_j\|^2\right)\right] && (4)\\
&= \sum_{i,j=1}^n \|T_{i,j}\|^2 && (5)
\end{aligned}$$

Taking square roots now yields the result.

1.2 Let $\pi : M_n(B(\mathcal{H})) \to B(\mathcal{H}^{(n)})$ be the identification given in the text.

i) Verify that π is a one-to-one, $*$-homomorphism.

ii) Let $E_j : \mathcal{H} \to \mathcal{H}^{(n)}$ be the map defined by $E_j(h)$ equal to the vector that has h for its j-th entry and is 0 elsewhere. Show that $E_j^* : \mathcal{H}^{(n)} \to \mathcal{H}$ is the map which sends a vector in $\mathcal{H}^{(n)}$ to its j-th component.

iii) Given $T \in B(\mathcal{H}^{(n)})$ set $T_{ij} = E_i^* T E_j$. Show that $\pi((T_{ij})) = T$ and that consequently π is onto.

Solution: Note that,

$$\sum_{k=1}^n E_k h_k = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}. \tag{6}$$

i) Let $(S_{i,j})$ and $(T_{i,j})$ be in $M_n(B(\mathcal{H}))$, let $h = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}, f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \in \mathcal{H}^{(n)}$ and $\alpha \in \mathbb{C}$. We have,

$$\begin{aligned}
\pi(\alpha(S_{i,j}) + (T_{i,j}))h &= \pi((\alpha S_{i,j} + T_{i,j}))h = \begin{bmatrix} \sum_{j=1}^n (\alpha S_{1,j} h_j + T_{1,j} h_j) \\ \vdots \\ \sum_{j=1}^n (\alpha S_{n,j} h_j + T_{n,j} h_j) \end{bmatrix} && (7) \\
&= \alpha \begin{bmatrix} \sum_{j=1}^n S_{1,j} h_j \\ \vdots \\ \sum_{j=1}^n S_{n,j} h_j \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^n S_{1,j} h_j \\ \vdots \\ \sum_{j=1}^n S_{n,j} h_j \end{bmatrix} && (8) \\
&= (\alpha\pi((S_{i,j})) + \pi((T_{i,j})))h. && (9)
\end{aligned}$$

Thus π is linear.
Let $(R_{ij}) = (S_{i,j})(T_{i,j})$.

$$\begin{aligned}
\pi((R_{i,j}))h &= \begin{bmatrix} \sum_{j=1}^n R_{1,j} h_j \\ \vdots \\ \sum_{j=1}^n R_{n,j} h_j \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j,k=1}^n S_{1,k} T_{k,j} h_j \\ \vdots \\ \sum_{j,k=1}^n S_{n,k} T_{k,j} h_j \end{bmatrix} && (10)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[\pi((S_{i,j}))\pi((T_{i,j}))]h &= \pi((S_{i,j})) \begin{bmatrix} \sum_{j=1}^n T_{1,j} h_j \\ \vdots \\ \sum_{j=1}^n T_{n,j} h_j \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=1}^n S_{1,k} \left(\sum_{j=1}^n T_{k,j} h_j \right) \\ \vdots \\ \sum_{k=1}^n S_{n,k} \left(\sum_{j=1}^n T_{k,j} h_j \right) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j,k=1}^n S_{1,k} T_{k,j} h_j \\ \vdots \\ \sum_{j,k=1}^n S_{n,k} T_{k,j} h_j \end{bmatrix} && (11)
\end{aligned}$$

Comparison of (10) and (11) shows that π is a homomorphism.

We now compute the adjoint of $\pi((T_{i,j}))$.

$$\left\langle \pi((T_{i,j})) \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \sum_{j=1}^n T_{1,j} h_j \\ \vdots \\ \sum_{j=1}^n T_{n,j} h_j \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \right\rangle \tag{12}$$

$$= \sum_{i,j=1}^n \langle T_{i,j} h_j, f_i \rangle = \sum_{i,j=1}^n \left\langle h_j, T_{i,j}^* f_i \right\rangle \tag{13}$$

$$= \left\langle \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}, \begin{bmatrix} \sum_{i=1}^n T_{i,1}^* f_i \\ \vdots \\ \sum_{i=1}^n T_{i,n}^* f_i \end{bmatrix} \right\rangle \tag{14}$$

$$= \left\langle \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}, \pi((T_{j,i}^*)) \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \right\rangle. \tag{15}$$

This shows that $\pi((T_{i,j}))^* = \pi((T_{j,i}^*)) = \pi((T_{i,j})^*)$ and so π is a $*$-homomorphism. Suppose $\pi((T_{i,j})) = 0$. Then for every $k \in \{1, \ldots, n\}$ and $h \in \mathcal{H}$ we have

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \pi((T_{ij}))E_k(h) = \begin{bmatrix} T_{1,k} h \\ \vdots \\ T_{n,k} h \end{bmatrix}. \tag{16}$$

Hence $T_{i,j} h = 0$ for all $h \in \mathcal{H}$ and for all $i, j \in \{1, \ldots, n\}$. It follows that $(T_{i,j}) = 0$ and π is one-to-one.

ii) Let $h \in \mathcal{H}$ and $\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} \in \mathcal{H}^{(n)}$. The definition of the adjoint gives us

$$\left\langle E_j^* \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}, h \right\rangle = \left\langle \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}, E_j h \right\rangle = \langle h_j, h \rangle \tag{17}$$

Therefore, E_j^* is the map that sends $\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$ to h_j.

iii) We have,

$$\left\langle \pi((T_{i,j})) \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \sum_{j=1}^n T_{1,j} h_j \\ \vdots \\ \sum_{j=1}^n T_{n,j} h_j \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \right\rangle \tag{18}$$

$$= \sum_{i,j=1}^n \langle T_{i,j} h_j, f_i \rangle = \sum_{i,j=1}^n \langle E_i^* T E_j h_j, f_i \rangle \tag{19}$$

$$= \sum_{i,j=1}^n \langle T E_j h_j, E_i f_i \rangle = \left\langle \sum_{j=1}^n T E_j h_j, \sum_{i=1}^n E_i f_i \right\rangle \tag{20}$$

$$= \left\langle T(\sum_{j=1}^n E_j h_j), \sum_{i=1}^n E_i f_i \right\rangle = \langle Th, f \rangle . \tag{21}$$

Thus $\pi((T_{i,j})) = T$ and π is onto.

1.3 Let $(T_{i,j})$ be in $M_n(B(\mathcal{H}))$. Prove that $(T_{i,j})$ is a contraction if and only if for every choice of $2n$ unit vectors $x_1, \ldots, x_n, y_1, \ldots, y_n$ in $\mathcal{H}$, the scalar matrix $(\langle T_{i,j} x_j, y_i \rangle)$ is a contraction.

Solution: We use the fact that if T is a bounded operator on a Hilbert space $\mathcal{H}$, then T is a contraction if and only if $|\langle Th, k \rangle| \leq 1$ for all $h, k \in \mathcal{H}$ of unit length.

Suppose first that $(T_{i,j})$ is a contraction and that $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}, \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}$ are vectors of unit length in $\mathbb{C}^n$.

A short calculation:

$$\left\| \begin{bmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{bmatrix} \right\|^2 = \sum_{k=1}^n \|\lambda_k x_k\|^2 = \sum_{k=1}^n |\lambda_k|^2 \|x_k\|^2 = 1, \tag{22}$$

shows that $\begin{bmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{bmatrix}$ and $\begin{bmatrix} \mu_1 y_1 \\ \vdots \\ \mu_n y_n \end{bmatrix}$ are of unit length in $\mathcal{H}^{(n)}$. It follows that,

$$\left| \left\langle (\langle T_{i,j} x_j, y_i \rangle) \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}, \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} \right\rangle \right| = \left| \sum_{i,j=1}^n \langle T_{i,j} x_j, y_i \rangle \lambda_j \overline{\mu_i} \right|$$

$$= \left| \sum_{i,j=1}^n \langle T_{i,j}(\lambda_j x_j), \mu_i y_i \rangle \right| \tag{23}$$

$$= \left| \left\langle (T_{i,j}) \begin{bmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{bmatrix}, \begin{bmatrix} \mu_1 y_1 \\ \vdots \\ \mu_n y_n \end{bmatrix} \right\rangle \right| \leq 1 \tag{24}$$

and hence $(\langle T_{i,j} x_j, y_i \rangle)$ is a contraction in M_n.

Conversely, let $\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$ and $\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$ be unit vectors in $\mathcal{H}^{(n)}$. Choose unit vectors $x_1, \ldots, x_n, y_1, \ldots, y_n$ in $\mathcal{H}$ and scalars $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$ such that $\lambda_j x_j = h_j$ and $\mu_i y_i = f_i$. We have,

$$1 = \left\| \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} \right\|^2 = \sum_{k=1}^{n} \|\lambda_k x_k\|^2 = \sum_{k=1}^{n} |\lambda_k|^2 \|x_k\|^2 = \sum_{k=1}^{n} |\lambda_k|^2 . \tag{25}$$

Which proves that the vectors $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}, \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}$ are of unit length in $\mathbb{C}^n$. A calculation similar to the one in (24) proves that $(T_{i,j})$ is a contraction.

1.4 Let$(T_{i,j})$ be in $M_n(\mathcal{H}^{(n)})$. Prove that $(T_{i,j})$ is positive if and only if for every choice of n vectors $x_1, \ldots, x_n$ in $\mathcal{H}$, the scalar matrix $(\langle T_{i,j} x_j, x_i \rangle)$ is positive.

Solution: If $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \in \mathbb{C}^n$ and $x_1, \ldots, x_n$ are vectors in $\mathcal{H}$, then

$$\left\langle (\langle T_{i,j} x_j, x_i \rangle) \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}, \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \right\rangle = \left\langle (T_{i,j}) \begin{bmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{bmatrix}, \begin{bmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{bmatrix} \right\rangle \tag{26}$$

If $(T_{i,j})$ is positive, (26) shows that $(\langle T_{i,j} x_j, x_i \rangle)$ is positive. If $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathcal{H}^{(n)}$ and $(\langle T_{i,j} x_j, x_i \rangle)$ is positive then set $\lambda_k = 1$ for all $k \in \{1, \ldots, n\}$ in (26) to prove $(T_{i,j})$ is positive.

1.5 Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, and let $\pi : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism with $\pi(\mathbf{1}) = \mathbf{1}$. Show that π is completely positive and completely bounded and that $\|\pi\| = \|\pi_n\| = \|\pi\|_{cb} = 1$.

Solution: If π is a $*$-homomorphism with $\pi(\mathbf{1}) = \mathbf{1}$, then π maps invertible elements in $\mathcal{A}$ to invertible elements in $\mathcal{B}$. Therefore $\sigma(\pi(a)) \subseteq \sigma(a)$ for any $a \in \mathcal{A}$. It follows that,

$$\begin{aligned} \|\pi(a)\|^2 &= \|\pi(a)^* \pi(a)\| && (27) \\ &= \|\pi(a^* a)\| = r(\pi(a^* a)) && (28) \\ &\leq r(a^* a) = \|a^* a\| = \|a\|^2 . && (29) \end{aligned}$$

where $r(x)$ denotes the spectral radius of x. Thus, $\|\pi\| \leq 1$. Since $\|\pi(\mathbf{1})\| = \|\mathbf{1}\| = 1$, we have $\|\pi\| = 1$. If a is positive, then there exists an $x \in \mathcal{A}$ such that $a = x^* x$. Therefore,

$$\pi(a) = \pi(x^* x) = \pi(x)^* \pi(x) \geq 0. \tag{30}$$

This shows that π is positive.

We have proved that a $*$-homomorphism that maps the identity to the identity is a norm 1, positive map. The definitions of addition and multiplication in $M_n(\mathcal{A})$ and the definition of π_n imply that π_n is indeed a unital, $*$-homomorphism. Hence π is completely positive and

$$\|\pi\|_{cb} = \sup\{\|\pi_n\| : n \geq 1\} = 1 = \|\pi\|, \tag{31}$$

shows that π is completely bounded.

1.6 Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be C^*-algebras, and let $\varphi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{C}$ be (completely) positive maps. Show that $\psi \circ \varphi$ is (completely) positive.

Solution: We first prove that $\psi \circ \varphi$ is positive. Let $a \in \mathcal{A}$ be positive. Since φ is a positive map $\varphi(a)$ is positive in $\mathcal{B}$. The fact that ψ is a positive map shows that $(\psi \circ \varphi)(a) = \psi(\varphi(a)) \geq 0$, proving our claim.

It is straightforward to check that $(\psi \circ \varphi)_n = \psi_n \circ \varphi_n$. This fact and a similar argument to the one used in the preceeding paragraph shows us that $\psi \circ \varphi$ is completely positive when φ and ψ are completely positive.

1.7 Let $\{E_{i,j}\}_{i,j=1}^n$ be matrix units for M_n, let $A = (E_{j,i})_{i,j=1}^n$ and let $B = (E_{i,j})_{i,j=1}^n$ be in $M_n(M_n)$. Show that A is unitary and that $\dfrac{1}{n}B$ is a rank one projection.

Solution: We begin by noting the following facts: $E_{i,j}^* = E_{j,i}$, $\sum_{k=1}^n E_{k,k} = \mathbf{1}_n$, the $n \times n$ identity matrix, and

$$E_{i,j}E_{p,q} = \begin{cases} 0 & \text{if } j \neq p \\ E_{i,q} & \text{if } j = p \end{cases}. \tag{32}$$

We have $A^* = (E_{j,i})^* = (E_{i,j}^*) = (E_{j,i}) = A$, and so A is self-adjoint. Thus $AA^* = A^2 = A^*A$. When we compute the (i,j)-th entry of A^2 we get,

$$\sum_{k=1}^n E_{k,i}E_{j,k} = \begin{cases} 0 & \text{if } i \neq j \\ \sum_{k=1}^n E_{kk} = \mathbf{1}_n & \text{if } i = j \end{cases} \tag{33}$$

From which it follows that $A^2 = \mathbf{1}$ and A is unitary.

Since $B^* = (E_{i,j})^* = (E_{j,i}^*) = (E_{i,j}) = B$, $\frac{1}{n}B$ is self-adjoint. The (i,j)-th entry of B^2 is

$$\sum_{k=1}^n E_{i,k}E_{k,j} = \sum_{k=1}^n E_{i,j} = nE_{i,j}. \tag{34}$$

Therefore the (i,j)-th entry of $\frac{1}{n^2}B^2$ is $\frac{1}{n}E_{i,j}$ and so $\left(\frac{1}{n}B\right)^2 = \frac{1}{n}B$. Thus $\frac{1}{n}B$ is a projection. It remains to be shown that B has rank one. The identification of $M_n(M_n)$ with M_{n^2} allows us to treat B as an operator on $\mathbb{C}^{n^2} = \underbrace{\mathbb{C}^n \oplus \ldots \oplus \mathbb{C}^n}_{n \text{ copies}}$. We compute the trace of B and note for a projection P that $\operatorname{rank} P = \operatorname{trace} P$. Note that the diagonal entries of B are 0 except the $((k-1)n+k, (k-1)n+k)$ entries for $k = 1, \ldots, n$ and so $\operatorname{trace} B = n$ which shows that $\operatorname{trace} \frac{1}{n}B = 1$. We now compute the

range of B. Let $h_k = \begin{bmatrix} \lambda_{k,1} \\ \vdots \\ \lambda_{k,n} \end{bmatrix} \in \mathbb{C}^n$. Now,

$$B(h_1 \oplus \ldots \oplus h_n) = \left(\sum_{j=1}^{n} E_{1,j} h_j \right) \oplus \ldots \oplus \left(\sum_{j=1}^{n} E_{n,j} h_j \right) = (\lambda_{1,1} + \ldots + \lambda_{n,n})(e_1 \oplus \ldots \oplus e_n). \tag{35}$$

Which shows that B, and hence $\frac{1}{n}B$, is the projection onto the span of the n^2-tuple whose entries are equal to 1 in the 1st, $(n+1)$-th, $\ldots$,$((n-1)n+1)$-th place and is 0 otherwise.

1.8 Let $\{E_{i,j}\}_{i,j=1}^n$ be a system of matrix units for $B(\mathcal{H})$, let $A = (E_{j,i})_{i,j=1}^n$ and let $B = (E_{i,j})_{i,j=1}^n$ be in $M_n(B(\mathcal{H}))$. Show that A is a partial isometry and that $\dfrac{1}{n}B$ is a rank one projection. Show that $\varphi_n(A) = B$ and $\|\varphi_n(A)\| = n$.

Solution: φ is the transpose map. We begin by noting the following facts: $E_{i,j}^* = E_{j,i}$,

$$E_{i,j}E_{l,m} = \begin{cases} 0 & \text{if } j \neq l \\ E_{i,m} & \text{if } j = l \end{cases}. \tag{36}$$

and

$$\sum_{k=1}^{n} E_{k,k} = \begin{bmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{bmatrix} = P_n, \tag{37}$$

where $\mathbf{1}_n$ denotes the $n \times n$ identity matrix.

A calculation similar to that in excercise 1.7 shows that A is self-adjoint and that

$$AA^* = A^2 = \begin{bmatrix} P_n & 0 & \cdots & 0 \\ 0 & P_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_n \end{bmatrix} \tag{38}$$

which is the projection of $\mathcal{H}^{(n)}$ onto the subspace spanned by vectors of the form $\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$ where all except possibly the first n entries of each h_k are 0. Therefore A is a partial isometry.

A similar argument to the one for Exercise 1.7 shows that $\frac{1}{n}B$ is a non-zero projection. We have,

$$\varphi_n(A) = \varphi_n((E_{j,i})) = (E_{j,i}^t) = (E_{i,j}) = B, \tag{39}$$

and

$$\|\varphi_n(A)\| = \|B\| = n \left\| \frac{1}{n} B \right\| = n. \tag{40}$$

1.9 Let A be in M_n and let A^t denote the transpose of A. Prove that A is positive if and only if A^t is positive and that $\|A\| = \|A^t\|$. Prove that these same results hold for operators on separable, infinite dimensional Hilbert space, when we fix an orthonormal basis, regard operators as infinite matrices and use this to define the transpose.

Solution: We prove the second part of the exercise which concerns Hilbert space, the first part being a special case of the second. To see that A^t is positive we write $A = B^*B$ for some $B \in B(\mathcal{H})$. Thus,

$$A^t = (B^*B)^t = B^t(B^*)^t = B^t(B^t)^* \geq 0. \tag{41}$$

By interchanging the roles of A and A^t we see that A is positive whenever A^t is positive.

Since $\mathcal{H}$ is a separable, infinite-dimensional Hilbert space we can assume that $\mathcal{H} = \ell^2$ with the orthonormal basis $\{e_n\}_{n\geq 1}$, where e_n is the sequence whose n-th entry is 1 and whose other entries are 0. Let $(\alpha_{i,j})$ be the matrix of A with respect to this orthonormal basis and let $(x_1, x_2, \ldots)$ be an element of $\mathcal{H}$. Denote by $\overline{A}$ and $\overline{x}$ the matrix $(\overline{\alpha}_{i,j})$ and the sequence $(\overline{x}_1, \overline{x}_2, \ldots)$. Note that $\|x\| = \|\overline{x}\|$. It follows from the definition of the inner-product that

$$\langle x, y\rangle = \overline{\langle \overline{x}, \overline{y}\rangle}. \tag{42}$$

It is also straightforward to show that $\overline{Ax} = \overline{A}\overline{x}$.

Let $\|x\| \leq 1$ and consider,

$$\left\|\overline{A}x\right\|^2 = \left\langle \overline{A}x, \overline{A}x\right\rangle = \overline{\langle A\overline{x}, A\overline{x}\rangle} = \|A\overline{x}\|^2 \leq \|A\|^2. \tag{43}$$

Therefore $\left\|\overline{A}\right\| \leq \|A\|$. By interchanging the roles of A and $\overline{A}$ we see that $\|A\| = \|\overline{\overline{A}}\| \leq \left\|\overline{A}\right\|$. Therefore $\|A\| = \left\|\overline{A}\right\|$. Since $\overline{A^t} = A^*$ we get,

$$\left\|A^t\right\| = \|\overline{A^t}\| = \|A^*\| = \|A\|. \tag{44}$$

The idea in the last two paragraphs gives an alternate proof of the positivity result. A short calculation,

$$\langle Ax, x\rangle = \langle x, A^*x\rangle = \left\langle x, \overline{A^t}x\right\rangle = \left\langle x, \overline{A^t\overline{x}}\right\rangle = \overline{\langle \overline{x}, A^t\overline{x}\rangle}, \tag{45}$$

shows that A is positive if and only if A^t is positive.

1.10 Prove that the map $\pi : M_n(\mathcal{A}) \to \mathcal{A} \otimes M_n$ defined by $\pi((a_{i,j})) = \sum_{i,j=1}^n a_{i,j} \otimes E_{i,j}$ is an algebra isomorphism.

Solution: Let α be a scalar. The addition and scalar multiplication of tensors satisfies $(a \otimes A) + (a \otimes B) = a \otimes (A+B)$ and $\alpha(a \otimes A) = (\alpha a) \otimes A = a \otimes (\alpha A)$ and therefore π is linear. Let $(c_{i,j}) = (a_{i,j})(b_{i,j})$. We have,

$$\pi((c_{i,j})) = \sum_{i,j=1}^n c_{i,j} \otimes E_{i,j} = \sum_{i,j,k=1}^n a_{i,k}b_{k,j} \otimes E_{i,j}. \tag{46}$$

Now,

$$\begin{aligned}
\pi((a_{i,j}))\pi((b_{i,j})) &= \left(\sum_{i,j=1}^n a_{i,j} \otimes E_{i,j}\right)\left(\sum_{k,l=1}^n a_{k,l} \otimes E_{k,l}\right) && (47)\\
&= \sum_{i,j,k,l=1}^n a_{i,j}b_{k,l} \otimes E_{i,j}E_{k,l} = \sum_{i,l=1}^n \sum_{j,k=1}^n a_{i,j}b_{k,l} \otimes E_{i,j}E_{k,l} && (48)\\
&= \sum_{i,l=1} \sum_{j=1} a_{i,j}b_{j,l} \otimes E_{i,l} \qquad \text{by using (33)}, && (49)
\end{aligned}$$

which (after relabeling the indices) is equal to (46). Thus π is a homomorphism. The involution in $\mathcal{A} \otimes M_n$ is defined by $(a \otimes A)^* = a^* \otimes A^*$ which proves that π is $*$-homomophism. We have,

$$\pi(\mathbf{1}) = \pi\left(\begin{bmatrix} \mathbf{1} & 0 & \cdots & 0 \\ 0 & \mathbf{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} \end{bmatrix}\right) = \sum_{i=1} \mathbf{1} \otimes E_{i,i} = \mathbf{1} \otimes \sum_{i=1} E_{i,i} = \mathbf{1} \otimes \mathbf{1}, \tag{50}$$

which shows that π is unital.

It remains to show π is bijective. Suppose that $\pi((a_{i,j})) = 0$, we want to show that $a_{k,l} = 0$ for all $k, l \in \{1, \ldots, n\}$. We compute,

$$\begin{aligned} (\mathbf{1} \otimes E_{k,k})\pi((a_{i,j}))(\mathbf{1} \otimes E_{l,l}) &= (\mathbf{1} \otimes E_{k,k}) \left(\sum_{i,j=1}^{n} a_{i,j} \otimes E_{i,j}\right) (\mathbf{1} \otimes E_{l,l}) && (51) \\ &= (\mathbf{1} \otimes E_{k,k}) \left(\sum_{i=1}^{n} a_{i,l} \otimes E_{i,l}\right) && (52) \\ &= \sum_{i=1}^{n} a_{i,l} \otimes E_{k,k}E_{i,l} = a_{k,l} \otimes E_{k,l} && (53) \end{aligned}$$

If we now use the fact that $\|a_{k,l} \otimes E_{k,l}\| = \|a_{k,l}\| \, \|E_{k,l}\|$ and $\|E_{k,l}\| = 1$ we see that $\|a_{k,l}\| = 0$ and so $a_{k,l} = 0$. Now suppose that $a \otimes A$ is an elementary tensor in $\mathcal{A} \otimes M_n$. If $A = (\alpha_{i,j})$, then we can write

$$a \otimes A = \sum_{i,j=1}^{n} a \otimes (\alpha_{i,j} E_{i,j}) = \sum_{i,j=1}^{n} (\alpha_{i,j} a) \otimes E_{i,j} = \pi((\alpha_{i,j} a)). \tag{54}$$

Thus, since π is linear, the range of π contains the span of the elementary tensors. The fact that π is a one-to-one, $*$-homomorpism implies that it is an isometry and has closed range. This together with the fact that the set of elementary tensors is dense in $\mathcal{A} \otimes M_n$ proves $\operatorname{range} \pi = \mathcal{A} \otimes M_n$ and so π is onto.