Completely Bounded Maps and Operator Algebras
Chapter 2: Positive Maps
Solutions to Exercises

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2.1 Let $\mathcal{S}$ be an operator system, $\mathcal{B}$ be a C^*-algebra and $\varphi : \mathcal{S} \to \mathcal{B}$ a postive map. Prove that φ is *self-sdoint*, i.e., that $\varphi(x^*) = \varphi(x)^*$.

Solution: We first prove the result when $x = x^*$. We can write $x = p_1 - p_2$ with p_1 and p_2 positive. Since φ is positive $\varphi(p_1)$ and $\varphi(p_2)$ are positive and so $\varphi(x) = \varphi(p_1) - \varphi(p_2)$, which is the difference of two postive elements, is self-adjoint.

Now let $x \in \mathcal{S}$ and write $x = x_1 + ix_2$ with x_1 and x_2 self-adjoint. Then we have,

$$\varphi(x^*) = \varphi(x_1 - ix_2) = \varphi(x_1) - i\varphi(x_2) = \varphi(x_1)^* - i\varphi(x_2)^* = (\varphi(x_1) + i\varphi(x_2))^* = \varphi(x)^*,$$

which completes the proof.

2.2 Let $\mathcal{S}$ be an operator system, $\mathcal{B}$ be a C^*-algebra and $\varphi : \mathcal{S} \to \mathcal{B}$ be a positive map. Prove that φ extends to a positive map on the norm closure of $\mathcal{S}$.

Solution: The positivity of φ implies that φ is bounded and in particular is uniformly continuous. It follows that φ extends in a unique way to a continuous map $\tilde{\varphi}$ on the norm closure of $\mathcal{S}$. It remains, therefore, to show that $\tilde{\varphi}$ is positive.

Let p be a positive element in $\overline{\mathcal{S}}$. Then, there is a sequence $\{x_n\}$ in $\mathcal{S}$ such that $\|p - x_n\| \to 0$ as $n \to \infty$. The fact that p is self-adjoint tells us that $\|p - x_n^*\| = \|p - x_n\|$ and so,

$$\left\| p - \frac{x_n + x_n^*}{2} \right\| \to 0 \text{ as } n \to \infty.$$

Set $h_n = (x_n + x_n^*)/2$ and note that h_n is self-adjoint. Let $\varepsilon > 0$. We claim that $h_n + \varepsilon\mathbf{1}$ is positive for sufficiently large values of n. We may assume that h_n is a sequence of operators on a Hilbert space $\mathcal{H}$. Let $x \in \mathcal{H}$ and choose N so large that $n \geq N$ implies $\|h_n - p\| < \varepsilon/2$. Then,

$$\langle (h_n + \varepsilon\mathbf{1})x, x\rangle = \langle (h_n + \varepsilon\mathbf{1} - (p + \varepsilon\mathbf{1}))x, x\rangle + \langle (p + \varepsilon\mathbf{1})x, x\rangle . \tag{1}$$

Since p is positive

$$\langle (p + \varepsilon\mathbf{1})x, x\rangle \geq \varepsilon \|x\|^2 , \tag{2}$$

and the Cauchy-Schwarz inequality gives,

$$|\langle (h_n + \varepsilon\mathbf{1} - (p + \varepsilon\mathbf{1}))x, x\rangle| \leq \|h_n + \varepsilon\mathbf{1} - (p + \varepsilon\mathbf{1})\| \|x\|^2 = \|h_n - p\| \|x\|^2 < \frac{\varepsilon}{2} \|x\|^2 . \tag{3}$$

Using (1), (2) and (3) we get,

$$\langle (h_n + \varepsilon\mathbf{1})x, x\rangle \geq (\varepsilon - \frac{\varepsilon}{2}) \|x\|^2 = \frac{\varepsilon}{2} \|x\|^2 ,$$

which establishes our claim.

Since $h_n + \varepsilon\mathbf{1} \to p + \varepsilon\mathbf{1}$ we have,

$$\tilde{\varphi}(p + \varepsilon\mathbf{1}) = \lim_{n\to\infty} \varphi(h_n + \varepsilon\mathbf{1}) \geq 0.$$

Thus,

$$\tilde{\varphi}(p) + \varepsilon\tilde{\varphi}(\mathbf{1}) \geq 0,$$

for every positive ε. When we let $\varepsilon \to 0$ we see that $\tilde{\varphi}(p) \geq 0$, which completes the proof.

2.3 Let $\mathcal{S}$ be an operator system and let $\varphi : \mathcal{S} \to \mathbb{C}$ be positive. Prove that $\|\varphi\| \leq \varphi(\mathbf{1})$.

Solution: Let $a \in \mathcal{S}$ and choose a unimodular $\lambda \in \mathbb{C}$ such that $|\varphi(a)| = \lambda\varphi(a) = \varphi(\lambda a)$. We have $|\varphi(a)| = \overline{\varphi(\lambda a)} = \varphi((\lambda a)^*)$, where the last equality follows from the fact that φ is positive. For any element a of a C^*-algebra $\operatorname{Re} a \leq \|a\| \mathbf{1}$. Thus,

$$\begin{aligned} |\varphi(a)| &= \frac{1}{2}(\varphi(\lambda a) + \varphi((\lambda a)^*)) = \varphi\left(\frac{\lambda a + (\lambda a)^*}{2}\right) \\ &= \varphi(\operatorname{Re}(\lambda a)) \leq \varphi(\|\lambda a\| \mathbf{1}) = \|a\| \varphi(\mathbf{1}). \end{aligned}$$

Therefore, $\|\varphi\| \leq \varphi(\mathbf{1})$.

2.4 Let $\mathcal{S}$ be an operator system and let $\varphi : \mathcal{S} \to C(X)$ where $C(X)$ denotes the continuous functions on a compact Hausdorff space X. Prove that if φ is positive, then $\|\varphi\| \leq \|\varphi(\mathbf{1})\|$.

Solution: For each x in X let $\pi_x : C(X) \to \mathbb{C}$ denote the function that maps f to $f(x)$. π_x is positive and so the map $\pi_x \circ \varphi : \mathcal{S} \to \mathbb{C}$ is positive. From Exercise 2.3 we get,

$$|\varphi(a)(x)| = |(\pi_x \circ \varphi)(a)| \leq ((\pi_x \circ \varphi)(\mathbf{1})) \|a\| = \varphi(\mathbf{1})(x) \|a\| \leq \|\varphi(\mathbf{1})\| \|a\|. \tag{4}$$

Thus, $\|\varphi(a)\| \leq \|\varphi(\mathbf{1})\| \|a\|$ and it follows that $\|\varphi\| \leq \|\varphi(\mathbf{1})\|$.

2.5 (Schwarz Inequality) Let $\mathcal{A}$ be a C^*-algebra and let $\varphi : \mathcal{A} \to \mathbb{C}$ be a positive linear functional. Prove that $|\varphi(x^*y)|^2 \leq \varphi(x^*x)\varphi(y^*y)$.

Solution: Let $t \in \mathbb{R}$ and choose a unimodular complex number λ such that $|\varphi(x^*y)| = \lambda\varphi(x^*y)$. Using the fact that φ is positive we get,

$$\begin{aligned} 0 &\leq \varphi((x+ty)^*(x+ty)) && (5) \\ &= \varphi(x^*x + t(x^*y + y^*x) + t^2y^*y) && (6) \\ &= \varphi(x^*x) + t(\varphi(x^*y) + \varphi(y^*x)) + t^2\varphi(y^*y) && (7) \\ &= \varphi(x^*x) + t(\varphi(x^*y) + \overline{\varphi(x^*y)}) + t^2\varphi(y^*y) && (8) \\ &= \varphi(x^*x) + 2\operatorname{Re}\varphi(x^*y)t + \varphi(y^*y)t^2 && (9) \end{aligned}$$

where (8) follows from the fact that φ is positive and therefore self-adjoint. The expression in (9) is a quadratic in t with real coefficients. Since this quadratic is always non-negative it follows that its discriminant is never positive. Thus,

$$(2\operatorname{Re}\varphi(x^*y))^2 \leq 4\varphi(x^*x)\varphi(y^*y). \tag{10}$$

By replacing x by λx and dividing out the factor of 4 we get,

$$|\varphi(x^*y)|^2 \leq \varphi(x^*x)\varphi(y^*y). \tag{11}$$

2.6 Let T be an operator on a Hilbert space $\mathcal{H}$, the *numerical radius* of T is defined by

$$w(T) = \sup\{|\langle Tx, x\rangle| : \ x \in \mathcal{H}, \|x\| = 1\}.$$

Prove that if $\varphi : \mathcal{S} \to B(\mathcal{H})$ is positive and $\varphi(\mathbf{1}) = \mathbf{1}$, then $w(\varphi(a)) \leq \|a\|$.

Solution: Let $a \in \mathcal{S}$ and let x be a unit vector in $\mathcal{H}$. Note that the linear functional ρ_x that maps an operator $T \in B(\mathcal{H})$ to $\langle Tx, x\rangle$ is positive. Therefore $\rho_x \circ \varphi : \mathcal{S} \to \mathbb{C}$ is positive. Hence by exercise 2.3,

$$|\langle \varphi(a)x, x\rangle| = |\rho_x \circ \varphi(a)| \leq (\rho_x \circ \varphi(\mathbf{1})) \|a\| = \rho_x(\mathbf{1}) \|a\| = \|x\|^2 \|a\| = \|a\| . \tag{12}$$

From which we see that $w(\varphi(a)) \leq \|a\|$.

2.7 Let T be an operator on a Hilbert space. Prove that $w(T) \leq 1$ if and only if $2 + (\lambda T) + (\lambda T)^* \geq 0$ for all complex numbers λ with $|\lambda| = 1$.

Solution: First note that $2 + (\lambda T) + (\lambda T)^*$ is positive if and only if $1 + \operatorname{Re} \lambda \langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. If $w(T) \leq 1$, then,

$$-\operatorname{Re} \lambda \langle Tx, x\rangle \leq |\lambda \langle Tx, x\rangle| = |\langle Tx, x\rangle| \leq w(T) \leq 1, \tag{13}$$

which proves that $2 + (\lambda T) + (\lambda T)^* \geq 0$.

For the converse choose a λ, (depending on x), of modulus 1 such that $\lambda \langle Tx, x\rangle = -|\langle Tx, x\rangle|$. We have,

$$|\langle Tx, x\rangle| = -\lambda \langle Tx, x\rangle = -\operatorname{Re} \lambda \langle Tx, x\rangle \leq 1, \tag{14}$$

and so $w(T) \leq 1$.

2.8 Prove that $w(T)$ defines a norm on $B(\mathcal{H})$, with $w(T) \leq \|T\| \leq 2w(T)$. Show that both inequalities are sharp.

Solution: $w(T)$ is defined as the supremum of a set of non-negative numbers and is therefore non-negative. Also $w(T) = 0$ if and only if $\langle Tx, x\rangle = 0$ for all x such that $\|x\| = 1$ if and only if $T = 0$. If $\alpha \in \mathbb{C}$, then

$$\begin{aligned} w(\alpha T) &= \sup\{|\langle \alpha Tx, x\rangle| : \ \|x\| = 1\} = \sup\{|\alpha| \, |\langle Tx, x\rangle| : \ \|x\| = 1\} && (15) \\ &= |\alpha| \sup\{|\langle Tx, x\rangle| : \ \|x\| = 1\} = |\alpha| \, w(T) && (16) \end{aligned}$$

If R, T are two operators and $\|x\| = 1$, then

$$|\langle (R + T)x, x\rangle| = |\langle Rx, x\rangle + \langle Tx, x\rangle| \leq |\langle Rx, x\rangle| + |\langle Tx, x\rangle| \leq w(R) + w(T) \tag{17}$$

implies that $w(R + T) \leq w(R) + w(T)$. This proves that $w(T)$ defines a norm on $B(\mathcal{H})$.

If $\|x\| = 1$, then an application of the Cauchy-Schwarz shows that,

$$|\langle Tx, x\rangle| \leq \|Tx\| \, \|x\| \leq \|T\| , \tag{18}$$

and so $w(T) \leq \|T\|$. For the other inequality begin by writing $T = A + iB$ where A and B are self-adjoint. Note that $w(T) = w(T^*)$. We use the fact that for a self-adjoint operator A, $\|A\| = \sup\{|\langle Ax, x\rangle| : \ \|x\| = 1\} = w(A)$. We have,

$$\begin{aligned} \|T\| &= \|A + iB\| \leq \|A\| + \|B\| && (19) \\ &= w(A) + w(B) = w\left(\frac{T + T^*}{2}\right) + w\left(\frac{T - T^*}{2i}\right) && (20) \\ &\leq \frac{w(T) + w(T^*) + w(T) + w(T^*)}{2} = 2w(T) && (21) \end{aligned}$$

The identity map $\mathbf{1}$ on any Hilbert space satisfies $w(\mathbf{1}) = 1 = \|\mathbf{1}\|$. To see that the best upper bound is 2 we consider the operator defined on $\mathbb{C}^2$ by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}$. We have,

$$\left\| T\begin{bmatrix} x \\ y \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} y \\ 0 \end{bmatrix} \right\|^2 = |y|^2 \leq |x|^2 + |y|^2 = \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2, \tag{22}$$

and so $\|T\| \leq 1$. By considering $T\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we see that $\|T\| = 1$. Now suppose that $\begin{bmatrix} x \\ y \end{bmatrix}$ has unit length and consider,

$$\left| \left\langle T\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle \right| = |yx| \leq \frac{|x|^2 + |y|^2}{2} = \frac{1}{2}. \tag{23}$$

Hence,

$$\|T\| \leq 2w(T) \leq 1 = \|T\|, \tag{24}$$

which shows that the other inequality is sharp.

2.9 Let $\mathcal{S}$ be an operator system, $\mathcal{B}$ a C^*-algebra and $\varphi : \mathcal{S} \to \mathcal{B}$ a linear map such that $\varphi(\mathbf{1})$ is positive, $\|\varphi(\mathbf{1})\| = \|\varphi\|$. Give an example to show that φ need not be positive. In similar vein, show that if $\mathcal{M}$ is as in proposition 2.12, $\varphi(\mathbf{1})$ is positive, with $\|\varphi(\mathbf{1})\| = \|\varphi\|$, then $\tilde{\varphi}$ need not be well-defined.

Solution: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in M_2 \tag{25}$$

and define $\varphi : M_2 \to M_2$ by

$$\varphi(A) = JA = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}. \tag{26}$$

Since φ is left multiplication by J, we have $\|\varphi\| = \|\varphi(\mathbf{1})\| = 2$. Further $\varphi(\mathbf{1}) = J \geq 0$. To see that φ is not positive consider

$$\varphi\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \tag{27}$$

which is not self-adjoint.

For the second part let $\mathcal{M}$ be the set of 2×2 upper triangular matrices and restrict the map φ in the previous paragraph to $\mathcal{M}$. Note that if $a \in \mathcal{M}$ is self-adjoint but $\varphi(a) \neq \varphi(a)^*$, then $\tilde{\varphi}$ is not well-defined. Consider the self-adjoint matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \tag{28}$$

and note that

$$\varphi\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, \tag{29}$$

which is not self-adjoint.

2.10 (Krein) Let $\mathcal{S}$ be an operator system contained in the C^*-algebra $\mathcal{A}$, and let $\varphi : \mathcal{S} \to \mathbb{C}$ be positive. Prove that φ can be extended to a positive map on $\mathcal{A}$.

Solution: The fact that φ is positive tells us that $\varphi(\mathbf{1})$ is a non-negative real number. If $\varphi(\mathbf{1}) = 0$ then $\|\varphi\| \leq \varphi(\mathbf{1}) = 0$ and this map extends to the zero functional. Assume that $\varphi(\mathbf{1}) \neq 0$ and let $\psi(a) = \varphi(\mathbf{1})^{-1}\varphi(a)$ and note that ψ is a positive map on $\mathcal{S}$ with $\|\psi\| = \psi(\mathbf{1}) = 1$. By the Hahn-Banach theorem ψ extends to a map $\tilde{\psi}$ on $\mathcal{A}$ with $\|\tilde{\psi}\| = 1$. $\tilde{\psi}$ is a unital contraction and is therefore a positive map. The map $\tilde{\varphi}$ defined by $\tilde{\varphi}(a) = \varphi(\mathbf{1})\tilde{\psi}(a)$ is a positive map which extends φ.

2.15 In this exercise we give an alternate proof of von Neumann's inequality. We assume that the reader has some familiarity with integration of operator valued functions. Let $T \in B(\mathcal{H})$ with $\|T\| < 1$ and, let p and q denote arbitrary polynomials.

(a) Let $P(t,T) = (\mathbf{1} - e^{-it}T)^{-1} + (\mathbf{1} - e^{it}T^*)^{-1} - \mathbf{1}$, and show that $P(t,T) \geq 0$ for all t.

(b) Show that $p(T) + q(T)^* = \dfrac{1}{2\pi}\int_0^{2\pi}(p(e^{it}) + \overline{q(e^{it})})P(t,T)\,dt$.

(c) Deduce von Neumann's inequality.

Solution:

(a) We know that if $\|T\| < 1$, then $\mathbf{1} - e^{-it}T$ is invertible. Let $R = e^{-it}T$ and note that the adjoint of $S = (\mathbf{1} - R)^{-1}$ is $S^* = (\mathbf{1} - R^*)^{-1}$. Let $h \in \mathcal{H}$ and set $k = Sh$. Then,

$$\begin{aligned}
\langle (S + S^* - \mathbf{1})h, h\rangle &= \langle Sh, h\rangle + \langle S^*h, h\rangle - \langle h, h\rangle && (30)\\
&= \left\langle k, S^{-1}k\right\rangle + \left\langle S^{-1}k, k\right\rangle - \left\langle S^{-1}k, S^{-1}k\right\rangle && (31)\\
&= \langle k, (\mathbf{1} - R)k\rangle + \langle (\mathbf{1} - R)k, k\rangle - \langle (\mathbf{1} - R)k, (\mathbf{1} - R)k\rangle && (32)\\
&= \|k\|^2 - \|Rk\|^2 \geq 0, && (33)
\end{aligned}$$

since R is a contraction.

(b) We know that if $\|T\| < 1$, then the operator $\mathbf{1} - e^{-it}T$ is invertible with inverse given by,

$$(\mathbf{1} - e^{-it}T)^{-1} = \sum_{n=0}^{\infty} e^{-int}T^n.$$

Let $p(z) = \alpha_0 + \ldots + \alpha_k z^k$. We have,

$$\int_0^{2\pi} p(e^{it})P(t,T)\,dt = \int_0^{2\pi} p(e^{it})(\mathbf{1} - e^{-it}T)^{-1}\,dt + \int_0^{2\pi} p(e^{it})(\mathbf{1} - e^{it}T^*)^{-1}\,dt - \int_0^{2\pi} p(e^{it})\mathbf{1}\,dt \tag{34}$$

Consider the first term on the right side of (34),

$$\begin{aligned}
\int_0^{2\pi} p(e^{it})(\mathbf{1} - e^{-it}T)^{-1}\,dt &= \int_0^{2\pi} p(e^{it})\left(\sum_{n=0}^{\infty} e^{-int}T^n\right)dt && (35)\\
&= \sum_{n=0}^{\infty}\int_0^{2\pi} p(e^{it})e^{-int}T^n\,dt && (36)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{k}\int_0^{2\pi} \alpha_m e^{imt}e^{-int}T^n\,dt && (37)\\
&= 2\pi(\alpha_0\mathbf{1} + \alpha_1 T + \ldots \alpha_k T^k) = 2\pi p(T) && (38)
\end{aligned}$$

A similar calculation shows that,

$$\int_0^{2\pi} p(e^{it})(\mathbf{1} - e^{it}T^*)^{-1}\,dt = \int_0^{2\pi} p(e^{it})\left(\sum_{n=0}^{\infty} e^{int}T^{*n}\right)dt = \sum_{n=0}^{\infty}\int_0^{2\pi} p(e^{it})e^{int}T^{*n}\,dt \quad (39)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{k}\int_0^{2\pi} \alpha_m e^{imt}e^{int}T^{*n}\,dt = 2\pi\alpha_0\mathbf{1} \quad (40)$$

and

$$\int_0^{2\pi} p(e^{it})\mathbf{1}\,dt = \int_0^{2\pi}\sum_{m=0}^{k}\alpha_k e^{it}\mathbf{1}\,dt = 2\pi\alpha_0\mathbf{1}. \quad (41)$$

Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi} p(e^{it})P(t,T)\,dt = p(T). \quad (42)$$

The same argument can be used to show that

$$q(T)^* = \frac{1}{2\pi}\int_0^{2\pi}\overline{q(e^{it})}P(t,T)\,dt. \quad (43)$$

(c) Suppose that $p + \overline{q}$ is positive, then the integral

$$\frac{1}{2\pi}\int_0^{2\pi}\left(p(e^{it}) + \overline{q(e^{it})}\right)P(t,T)\,dt,$$

is positive, being a limit of Riemann sums, each of which is a positive operator. This last statement follows from the fact that $P(t,T)$ and $p+\overline{q}$ are positive. This proves Theorem 2.6 and von Neumann's inequality follows for operators T with $\|T\| < 1$. We now prove the case $\|T\| = 1$. Let $0 < r < 1$ and note that $\|rT\| < 1$. Thus for any polynomial p, $\|p(rT)\| \le \|p\|_\infty$. Therefore,

$$\|p(T)\| = \lim_{r\to 1^-}\|p(rT)\| \le \|p\|_\infty. \quad (44)$$

2.16 (Wermer) In this exercise we give an alternate proof of von Neumann's inequality which is only valid for matrices. We assume that the reader is familiar with the singular value decomposition of a matrix. Let $T \in M_n$ with $\|T\| \le 1$ and write $T = USV$ with U, V unitary and $S = \mathrm{diag}(s_1, \ldots, s_n)$ a positive diagonal matrix, $0 \le s_i \le 1$, $i = 1, \ldots, n$. Define an analytic matrix-valued function $T(z_1, \ldots, z_n) = UZV$ where $Z = \mathrm{diag}(z_1, \ldots, z_n)$, $|z_i| \le 1$, $i = 1, \ldots, n$. Fix a polynomial p.

i) Let $x, y \in \mathbb{C}^n$ and let $f(z_1, \ldots, z_n) = \langle p(T(z_1, \ldots, z_n))x, y\rangle$. Deduce that f achieves its maximum modulus at a point where $|z_1| = \ldots = |z_n| = 1$. Note that at such a point $T(z_1, \ldots, z_n)$ is unitary.

ii) Deduce that $\sup_{|z_i|\le 1}\|p(T(z_1, \ldots, z_n))\|$ is attained at a point where $T(z_1, \ldots, z_n)$ is unitary.

iii) Deduce that $\|p(T)\| \le \sup\|p(W)\|$ over $W \in M_n$, unitary.

iv) Show that for W unitary $\|p(W)\| \le \|p\|_\infty$.

v) Deduce von Neumann's inequality for $T \in M_n$.

Solution:

i) Note that f is an analytic function. We have

$$\sup_{|z_i|\leq 1} |f(z_1,\ldots,z_n)| = \sup_{|z_i|\leq 1, i\neq 1} \sup_{|z_1|\leq 1} |f(z_1,\ldots,z_n)| \tag{45}$$

$$= \sup_{|z_i|\leq 1, i\neq 1} |f(w_1,z_2,\ldots,z_n)|, \tag{46}$$

where w_1 is a complex number of modulus 1, by the maximum modulus principle. Repeating this argument we see that f achieves its maximum modulus at a point where $|z_i| = 1$ for all $i = 1,\ldots,n$. In this case the matrix Z is diagonal matrix with diagonal entries of absolute value 1, thus Z, and consequently $T(z_1,\ldots,z_n) = UZV$ is unitary.

ii) We have,

$$\sup_{|z_i|\leq 1} \|p(T(z_1,\ldots,z_n))\| = \sup_{|z_i|\leq 1} \sup_{x,y\in\mathbb{C}^n} |\langle p(T(z_1,\ldots,z_n))x, y\rangle|$$
$$= \sup_{x,y\in\mathbb{C}^n} \sup_{|z_i|\leq 1} |f(z_1,\ldots,z_n)|,$$

from which it follows by i) that the supremum occurs at a point where $T(z_1,\ldots,z_n)$ is unitary.

iii) Since the singular values of T satisfy $0 \leq s_i \leq 1$, we have,

$$\|p(T)\| = \|p(T(s_1,\ldots,s_n))\| \leq \sup_{|z_i|\leq 1} \|p(T(z_1,\ldots,z_n))\| \leq \sup \|p(W)\|.$$

where the last inequality follows from part ii).

iv) If W is unitary, then $p(W)$ is normal and so $\|p(W)\| = r(p(W))$ where r denotes the spectral radius. Let $\lambda \in \sigma(W)$. Since the eigenvalues of W have modulus 1, $|p(\lambda)| \leq \sup_{|z|=1} |p(z)| = \|p\|_\infty$, and so

$$\|p(W)\| = \sup\{|p(\lambda)| : \lambda \in W\} \leq \|p\|_\infty.$$

v) By combining the results from parts iii) and iv) we get

$$\|p(T)\| \leq \sup \|p(W)\| \leq \|p\|_\infty.$$

2.20 (Korovkin) Let $f \in C([0,1])$ and let $g_x(t) = (t-x)^2$.

i) Given $\varepsilon > 0$, show that there exists a constant $c > 0$ depending only on ε and f such that

$$|f(t) - f(x)| \leq \varepsilon + cg_x(t) \text{ for all } 0 \leq x, t \leq 1.$$

ii) Let $\varphi : C([0,1]) \to C([0,1])$ be a positive map with $\varphi(\mathbf{1}) = \mathbf{1}$. Show that

$$-\varepsilon - c\varphi(g_x)(x) \leq \varphi(f)(x) - f(x) \leq \varepsilon + c\varphi(g_x)(x),$$

and deduce that $\|\varphi(f) - f\| \leq \varepsilon + c\sup_x |\varphi(g_x)(x)|$.

iii) Let $\varphi_n : C([0,1]) \to C([0,1])$ be a sequence of positive maps. Prove that if $\|\varphi_n(f_i) - f_i\| \to 0$ as $n \to \infty$ for $f_i(t) = t^i$, $i = 0,1,2$, then $\|\varphi_n(f) - f\| \to 0$ as $n \to \infty$ for all $f \in C([0,1])$.

Solution:

i) Since f is uniformly continuous we may choose $\delta > 0$ such that $|f(t) - f(x)| < \varepsilon$ for $|t - x| < \delta$ with $t \in [0, 1]$. Let $M = \sup\{|f(t)| : t \in [0, 1]\}$. For $|t - x| < \delta$ we have $|f(t) - f(x)| \leq \varepsilon$ and for $|t - x| \geq \delta$ we have,

$$|f(t) - f(x)| \leq 2M \leq \frac{2M}{\delta^2} g_x(t) = c g_x(t).$$

Combining these inequalities we get,

$$|f(t) - f(x)| \leq \varepsilon + c g_x(t). \tag{47}$$

ii) From (47),

$$-\varepsilon \mathbf{1} - c g_x \leq f - f(x) \leq \varepsilon \mathbf{1} + c g_x. \tag{48}$$

Applying φ to both sides of (48) and noting that φ is unital yields

$$-\varepsilon - c\varphi(g_x)(x) \leq \varphi(f)(x) - f(x) \leq \varepsilon + c\varphi(g_x)(x).$$

Hence,

$$|\varphi(f)(x) - f(x)| \leq \varepsilon + c\,|\varphi(g_x)(x)|\,,$$

and taking the supremum over $x \in [0, 1]$ yields,

$$\|\varphi(f) - f\| \leq \varepsilon + c \sup_x |\varphi(g_x)(x)|\,.$$

iii) Note that,

$$\varphi_n(g_x) - g_x = \varphi_n(f_2) - f_2 - 2x(\varphi_n(f_1) - f_1) + x^2(\varphi_n(f_0) - f_0).$$

Therefore,

$$\|\varphi_n(g_x) - g_x\| \leq 6 \max_{i=0,1,2} \|\varphi_n(f_i) - f_i\| \leq \varepsilon$$

for sufficiently large values of n. Note that $g_x(x) = 0$ and so

$$|\varphi_n(g_x)(x)| = |\varphi_n(g_x)(x) - g_x(x)| \leq \varepsilon.$$

Thus

$$\|\varphi_n(f) - f\| \leq \varepsilon + c \sup_x |\varphi_n(g_x)(x)| \leq (1 + c)\varepsilon,$$

which shows $\varphi_n(f) \to f$ uniformly.

2.21 The *Bernstein maps* $\varphi_n : C([0, 1]) \to C([0, 1])$ are defined by

$$\varphi_n(f)(t) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) t^k (1 - t)^{n-k}.$$

i) Verify that the Bernstein maps are positive maps with $\varphi_n(\mathbf{1}) = \mathbf{1}$, $\varphi_n(t) = t$, $\varphi_n(t^2) = t^2 + \dfrac{t - t^2}{n}$.

ii) Deduce that $\|\varphi_n(f) - f\| \to 0$ for all $f \in C([0, 1])$.

iii) Deduce the Weierstrass theorem, i.e., prove that the polynomials are dense in $C([0, 1])$.

Solution:

i) Note that the quantity $t^k(1-t)^{n-k}$ is non-negative on $[0,1]$. Thus if $f \geq 0$, then,

$$\binom{n}{k} f\left(\frac{k}{n}\right) t^k(1-t)^{n-k},$$

is non-negative. This implies that $\varphi_n(f) \geq 0$.
Denote by f_i the functions $f_i(t) = t^i$. We have by the binomial theorem that,

$$\varphi_n(\mathbf{1})(t) = \sum_{k=0}^{n} \binom{n}{k} t^k(1-t)^{n-k} = (t + (1-t))^n = 1.$$

Now,

$$\begin{aligned}
\varphi_n(f_1)(t) &= \sum_{k=0}^{n} \binom{n}{k} f_1\left(\frac{k}{n}\right) t^k(1-t)^{n-k} && (49)\\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} t^k(1-t)^{n-k} && (50)\\
&= \sum_{k=1}^{n} \binom{n}{k} \frac{k}{n} t^k(1-t)^{n-k} && (51)\\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} t^k(1-t)^{n-k} && (52)\\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} t^{k+1}(1-t)^{n-1-k} && (53)\\
&= t\sum_{k=0}^{n-1} t^k(1-t)^{n-1-k} && (54)\\
&= t(t + (1-t))^{n-1} = t. && (55)
\end{aligned}$$

A similar calculation,

$$
\begin{aligned}
\varphi_n(f_2)(t) &= \sum_{k=0}^{n} \binom{n}{k} \frac{k^2}{n^2} t^k (1-t)^{n-k} && (56)\\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{k}{n} t^k (1-t)^{n-k} && (57)\\
&= t \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{k}{n} t^{k-1} (1-t)^{n-k} && (58)\\
&= t \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{k+1}{n} t^k (1-t)^{n-1-k} && (59)\\
&= t \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{k}{n} t^k (1-t)^{n-1-k} + t \sum_{k=0}^{n-1} \frac{1}{n} t^k (1-t)^{n-1-k} && (60)\\
&= \frac{n-1}{n} t \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{k}{n-1} t^k (1-t)^{n-1-k} + \frac{t}{n} \sum_{k=0}^{n-1} t^k (1-t)^{n-1-k} && (61)\\
&= \frac{n-1}{n} t^2 + \frac{1}{n} t = t^2 + \frac{t-t^2}{n}. && (62)
\end{aligned}
$$

ii) We will make use of the results from exercise 20. Note that $\|\varphi_n(f_0) - f_0\| = \|\varphi_n(f_1) - f_1\| = 0$ and $\|\varphi_n(f_2) - f_2\| = \left\| \frac{f_1 - f_2}{n} \right\| \leq \frac{2}{n} \to 0$ as $n \to \infty$. By the result in Exercise 20.iii this implies that $\|\varphi_n(f) - f\| \to 0$ for all $f \in C([0,1])$.

iii) From the part ii) of this exercise we know that the collection $\{\varphi_n(f) : f \in C([0,1])\}$ is dense in $C([0,1])$. This collection is contained in the set of polynomials and hence the polynomials are dense in $C([0,1])$.

2.22 A sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers is called a *Hausdorff moment sequence* if there exists a positive (finite) Borel measure μ on $[0,1]$ such that $a_n = \int_0^1 t^n \, d\mu(t)$ for all n. Set $b_{n,m} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_{k+m}$ for all $n, m \geq 0$.

i) Assuming the existence of such a measure μ, show that $b_{n,m} = \int_0^1 t^m (1-t)^n \, d\mu(t)$ and deduce that necessarily $b_{n,m} \geq 0$ for all $n, m \geq 0$.

ii) Let $\mathcal{P} \subseteq C([0,1])$ denote the set of polynomials and define $\varphi : \mathcal{P} \to \mathbb{C}$ by setting $\varphi(t^n) = a_n$. Show that if $b_{n,m} \geq 0$ for all $n, m \geq 0$ then φ is a positive map.

iii) Prove that $\{a_n\}_{n=0}^{\infty}$ is a Hausdorff moment sequence if and only if $b_{n,m} \geq 0$ for all $m, n \geq 0$.

Solution:

i) We have,

$$
\begin{aligned}
b_{n,m} &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_{k+m} \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \int_0^1 t^{k+m}\, d\mu(t) && (63)\\
&= \int_0^1 t^m \sum_{k=0}^{n} \binom{n}{k} (-1)^k t^k\, d\mu(t) && (64)\\
&= \int_0^1 t^m (1-t)^n\, d\mu(t) \geq 0. && (65)
\end{aligned}
$$

Since μ is positive and the function $f(t) = t^m(1-t)^n$ is non-negative on $[0,1]$.

ii) Let $\mathcal{P}_j$ denote the set of polynomials of degree at most j and let $f_j \in \mathcal{P}_j$ be defined by $f_j(t) = t^j$. We begin by proving that each of the Bernstein maps $\varphi_n : \mathcal{P}_j \to \mathcal{P}_j$. It is clear that $\varphi_n(\mathbf{1}) = \mathbf{1}$ and so the claim is true for $j = 0$. We proceed by induction. Assuming the result for all polynomials of degree $j-1$ or less we prove it for polynomials of degree j. Since φ_n is linear for all n we need only check that $\varphi_n(f_j) \in \mathcal{P}_j$. We have,

$$
\begin{aligned}
\varphi_n(f_j)(t) &= \sum_{k=0}^{n} \binom{n}{k} f_j\left(\frac{k}{n}\right) t^k (1-t)^{n-k} && (66)\\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{k^j}{n^j} t^k (1-t)^{n-k} && (67)\\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{k^{j-1}}{n^{j-1}} t^k (1-t)^{n-k} && (68)\\
&= t\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{k^{j-1}}{n^{j-1}} t^{k-1} (1-t)^{n-k} && (69)\\
&= t\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(k+1)^{j-1}}{n^{j-1}} t^k (1-t)^{n-1-k} && (70)\\
&= t\varphi_{n-1}(g_{n,j})(t) && (71)
\end{aligned}
$$

where $g_{n,j}(t) = \left(t + \frac{1}{n}\right)^{j-1} \in \mathcal{P}_{j-1}$. By the induction hypothesis $\varphi_{n-1}(g) \in \mathcal{P}_{j-1}$ and so $\varphi_n(f_j) = f_1\varphi_{n-1}(g_{n,j}) \in \mathcal{P}_j$.

We also have,

$$
\begin{aligned}
0 \leq b_{n,m} &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_{k+m} && (72)\\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \varphi(t^{k+m}) && (73)\\
&= \varphi\left(t^m \sum_{k=0}^{n} \binom{n}{k} (-1)^k t^k \right) && (74)\\
&= \varphi\left(t^m (1-t)^n\right) && (75)
\end{aligned}
$$

This shows that if $f \geq 0$ then,

$$(\varphi \circ \varphi_n(f))(t) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) \varphi(t^k (1-t)^{n-k}) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) b_{k,n-k} \geq 0.$$

Now assume that $f \in \mathcal{P}$ and suppose that f has degree j. We have seen that $\varphi_n(f) \in \mathcal{P}_j$ for all n. The map $\varphi|_{\mathcal{P}_j}$ being a linear map on a finite dimensional space must be continuous. Therefore,

$$\varphi(f) = \varphi(\lim_{n\to\infty} \varphi_n(f)) = \lim_{n\to\infty} \varphi(\varphi_n(f)) \geq 0.$$

iii) We have seen in part i) that if $\{a_n\}_{n=0}^{\infty}$ is a Hausdorff moment sequence, then $b_{n,m}$ is non-negative. For the converse assume that $b_{n,m}$ is positive for all $n, m \geq 0$ and note this implies that the linear functional $\varphi : \mathcal{P} \to \mathbb{C}$ is positive. By Exercise 2.2, φ has a positive extension to the closure of $\mathcal{P}$, which is $C([0,1])$. The Riesz representation theorem implies that there is a positive (finite) Borel measure such that

$$a_n = \varphi(t^n) = \int_0^1 t^n \, d\mu(t).$$