Completely Bounded Maps and Operator Algebras
Chapter 3: Completely Positive Maps
Solutions to Exercises

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3.1 Prove that $\|\varphi_n\| \leq \|\varphi_k\|$ for $n \leq k$ and that if φ_k is positive, then φ_n is positive.

Solution: Note that the map $\Phi : M_n(\mathcal{A}) \to M_k(\mathcal{A})$ defined by,

$$\Phi(A) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

is a one-to-one, $*$-homomorphism. This allows us to identify $M_n(\mathcal{A})$ as a C^*-subalgebra of $M_k(\mathcal{A})$. Under this identification φ_n is the compression, to $M_n(\mathcal{A})$, of φ_k. It follows that φ_n is positive whenever φ_k is positive and that $\|\varphi_n\| \leq \|\varphi_k\|$.

3.2 Let P, Q, A be operators on some Hilbert space $\mathcal{H}$ with P, Q positive.

i) Show that $\begin{bmatrix} P & A \\ A^* & Q \end{bmatrix} \geq 0$ if and only if $|\langle Ax, y\rangle|^2 \leq \langle Py, y\rangle \langle Qy, y\rangle$ for all x, y in $\mathcal{H}$.

ii) Prove that $\begin{bmatrix} \mathbf{1} & a \\ a^* & b \end{bmatrix}$ is positive in $M_2(\mathcal{A})$ if and only if $a^*a \leq b$.

iii) Show that if $\begin{bmatrix} P & A \\ A^* & Q \end{bmatrix} \geq 0$ then for any x in $\mathcal{H}$, we have that

$$0 \leq \langle (P + A^* + A + Q)x, x\rangle \leq (\sqrt{\langle Px, x\rangle} + \sqrt{\langle Qx, x\rangle})^2,$$

and hence $\|P + A + A^* + Q\| \leq (\|P\|^2 + \|Q\|^2)^{1/2}$.

iv) Show that if $\begin{bmatrix} P & A \\ A^* & P \end{bmatrix} \geq 0$, then $A^*A \leq \|P\|\, P$ and in particular $\|A\| \leq \|P\|$.

Solution:

i) Let $x, y \in \mathcal{H}$. We recall the fact (Exercise 1.3) that the operator matrix $((T_{i,j}))$ is positive if and only if for any choice of n vectors $x_1, \ldots, x_n$ the scalar matrix $(\langle T_{i,j}x_j, x_i\rangle)$ is positive. Using this we see that

$$\begin{bmatrix} P & A \\ A^* & Q \end{bmatrix}$$

is positive if and only if the matrix

$$\begin{bmatrix} \langle Py, y\rangle & \langle Ax, y\rangle \\ \langle A^*y, x\rangle & \langle Qx, x\rangle \end{bmatrix}. \tag{2}$$

A 2×2 scalar matrix is positive if and only if it has non-negative trace and determinant. Since P, Q are positive the matrix in (2) has positive trace. Thus, positivity of (2) is equivalent to the matrix having non-negative determinant. i.e.,

$$\langle Py, y\rangle \langle Qx, x\rangle \geq \langle Ax, y\rangle \langle A^*y, x\rangle = \langle Ax, y\rangle \langle y, Ax\rangle = |\langle Ax, y\rangle|^2.$$

ii) We can assume that $\mathcal{A}$ is a C^*-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. We know that

$$\begin{bmatrix} \mathbf{1} & A \\ A^* & B \end{bmatrix} \geq 0$$

if and only if $|\langle Ax, y\rangle|^2 \leq \langle y, y\rangle \langle Bx, x\rangle$. Setting $y = Ax$ yields

$$|\langle Ax, Ax\rangle|^2 \leq \langle Ax, Ax\rangle \langle Bx, x\rangle,$$

and so $\langle A^*Ax, x\rangle = \langle Ax, Ax\rangle \leq \langle Bx, x\rangle$, which proves that $A^*A \leq B$.
Conversely if we know that $\langle A^*Ax, x\rangle \leq \langle Bx, x\rangle$ then by multiplying both sides by $\langle y, y\rangle$ and using the Cauchy-Schwarz inequality we get,

$$|\langle Ax, y\rangle|^2 \leq \langle Ax, Ax\rangle \langle y, y\rangle \leq \langle Bx, x\rangle \langle y, y\rangle.$$

iii) Notice that,

$$\langle (P + A^* + A + Q)x, x\rangle = \left\langle \begin{bmatrix} P & A \\ A^* & Q \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \end{bmatrix} \right\rangle \geq 0.$$

For the other ineqaulity we use the fact that $|\langle Ax, x\rangle|^2 \leq \langle Px, x\rangle \langle Qx, x\rangle$. We have,

$$\begin{aligned} \langle (P + A^* + A + Q)x, x\rangle &= \langle Px, x\rangle + \langle A^*x, x\rangle + \langle Ax, x\rangle + \langle Qx, x\rangle && (3) \\ &\leq \langle Px, x\rangle + 2\sqrt{\langle Px, x\rangle}\sqrt{\langle Qx, x\rangle} + \langle Qx, x\rangle && (4) \\ &= (\sqrt{\langle Px, x\rangle} + \sqrt{\langle Qx, x\rangle})^2 && (5) \\ &\leq (\|P\|^{1/2}\|x\| + \|Q\|^{1/2}\|x\|)^2 \leq (\|P\|^{1/2} + \|Q\|^{1/2})^2 \|x\|^2. && (6) \end{aligned}$$

which proves the other inequality. Since $P + A + A^* + Q$ is self-adjoint the last inequality implies that $\|P + A + A^* + Q\| \leq (\|P\|^2 + \|Q\|^2)^{1/2}$.

iv) The positivity of

$$\begin{bmatrix} P & A \\ A^* & P \end{bmatrix}$$

gives,

$$|\langle Ax, Ax\rangle|^2 \leq \langle PAx, Ax\rangle \langle Px, x\rangle \leq \|P\| \langle Ax, Ax\rangle \langle Px, x\rangle.$$

It follows that $\langle Ax, Ax\rangle \leq \|P\| \langle Px, x\rangle$ which implies $A^*A \leq \|P\| P$ and $\|A\| \leq \|P\|$.

3.3 Prove a non-unital version of proposition 3.2.

Solution: Let $\mathcal{S}$ be an operator system, $\mathcal{B}$ a C^*-algebra and $\varphi : \mathcal{S} \to \mathcal{B}$ a 2-positive map. Let $a \in \mathcal{S}$ with $\|a\| \leq 1$. Since φ is 2-positive we have,

$$\varphi_2 \begin{bmatrix} \mathbf{1} & a \\ a^* & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \varphi(\mathbf{1}) & \varphi(a) \\ \varphi(a)^* & \varphi(\mathbf{1}) \end{bmatrix} \geq 0. \tag{7}$$

By 3.2.iii we have $\|\varphi(a)\| \leq \|\varphi(\mathbf{1})\|$.

3.4 (Modified Schwarz Inequality for 2-positive maps) Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, $\varphi : \mathcal{A} \to \mathcal{B}$ 2-positive. Prove that $\varphi(a)^*\varphi(a) \leq \|\varphi(\mathbf{1})\| \, \varphi(a^*a)$ and that $\|\varphi(a^*b)\|^2 \leq \|\varphi(a^*a)\| \, \|\varphi(b^*b)\|$.

Solution: If

$$\begin{bmatrix} P & A \\ A^* & Q \end{bmatrix}$$

is a positive operator matrix then,

$$\langle Ax, Ax\rangle^2 \leq \langle PAx, Ax\rangle \langle Qx, x\rangle \leq \|P\| \langle Ax, Ax\rangle \langle Qx, x\rangle ,$$

from which we get that $A^*A \leq \|P\| \, Q$ and $\|A\|^2 \leq \|P\| \, \|Q\|$.

Notice that the matrix,

$$\begin{bmatrix} \mathbf{1} & a \\ a^* & a^*a \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} \geq 0,$$

and since φ is 2-positive

$$\varphi\left(\begin{bmatrix} \mathbf{1} & a \\ a^* & a^*a \end{bmatrix}\right) = \begin{bmatrix} \varphi(\mathbf{1}) & \varphi(a) \\ \varphi(a)^* & \varphi(a^*a) \end{bmatrix} \geq 0.$$

From our observation in the first paragraph $\varphi(a)^*\varphi(a) \leq \|\varphi(\mathbf{1})\| \, \varphi(a^*a)$.

Next consider the matrix

$$\begin{bmatrix} a^*a & a^*b \\ b^*a & b^*b \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \geq 0.$$

The 2-positivity of φ implies that the matrix,

$$\begin{bmatrix} \varphi(a^*a) & \varphi(a^*b) \\ \varphi(b^*a) & \varphi(b^*b) \end{bmatrix} = \begin{bmatrix} \varphi(a^*a) & \varphi(a^*b) \\ \varphi(a^*b)^* & \varphi(b^*b) \end{bmatrix} \geq 0,$$

and so $\varphi(a^*b)^*\varphi(a^*b) \leq \|\varphi(a^*a)\| \, \varphi(b^*b)$ and $\|\varphi(a^*b)\|^2 \leq \|\varphi(a^*a)\| \, \|\varphi(b^*b)\|$.

3.5 Let $\mathcal{A}$ be a C^*-algebra with unit. Show that the maps $\mathrm{Tr}\,, \sigma : M_n(\mathcal{A}) \to \mathcal{A}$ defined by $\mathrm{Tr}\,((a_{i,j})) = \sum_{i=1}^n a_{i,i}$, and $\sigma((a_{i,j})) = \sum_{i,j=1}^n a_{i,j}$ are completely positive maps. Deduce that if $\|(a_{i,j})\| \leq 1$, then $\left\|\sum_{i,j=1}^n a_{i,j}\right\| \leq n$.

Solution: We identify $\mathcal{A}$ with a C^*-subalgebra of $B(\mathcal{H})$. Let E_j be the operator in $B(\mathcal{H}, \mathcal{H}^{(n)})$ defined by $E_j(h) = (0, \ldots, h, \ldots, 0)$ where the h appears in the j-th entry. Let $A = (A_{i,j})_{i,j=1}^n$ be an element of $M_n(B(\mathcal{H}))$. A calculation done in Exercise 1.2.iii shows that

$$E_i^* A E_j = A_{i,j}$$

and so

$$\mathrm{Tr}\,(A) = \sum_{i=1}^n A_{i,i} = \sum_{i=1}^n E_i^* A E_i,$$

which is completely positive.

Similarly,

$$\begin{aligned}\sigma(A) &= \sum_{i,j}^{n} E_i^* A E_j = \sum_{i=1}^{n}\sum_{j=1}^{n} E_i^* A E_j \\ &= (\sum_{i=1}^{n} E_i^*) A (\sum_{j=1}^{n} E_j) = (\sum_{i=1}^{n} E_i)^* A (\sum_{j=1}^{n} E_j),\end{aligned}$$

which proves that σ is completely positive.

3.6 (Choi) Let $\mathcal{A}$ be a C^*-algebra, let λ be a complex number with $|\lambda| = 1$, let U_λ be the unitary element of $M_n(\mathcal{A})$ that is diagonal with $u_{i,i} = \lambda^i \mathbf{1}$, and let $\mathrm{Diag} : M_n(\mathcal{A}) \to M_n(\mathcal{A})$ be defined by $\mathrm{Diag}((a_{i,j})) = (b_{i,j})$, where $b_{i,j} = 0$, for $i \neq j$ and $b_{i,i} = a_{i,i}$.

(i) Show that $U_\lambda^*(a_{i,j})U_\lambda = (\lambda^{j-i} a_{i,j})$.

(ii) By considering the non-trivial n-th roots of unity, show that the map $\Phi : M_n(\mathcal{A}) \to M_n(\mathcal{A})$ defined by $\Phi(A) = n\mathrm{Diag}(A) - A$ is completely positive.

(iii) Show that the map $\Phi : M_n \to M_n$ defined by $\Phi(A) = (n-1)\mathrm{Diag}(A) - A$ is not positive.

Solution:

(i) Recall that if $D = \mathrm{diag}(d_1, \ldots, d_n)$ then, $D(a_{i,j}) = (d_i a_{i,j})$ and $(a_{ij})D = (a_{i,j} d_j)$. Note that if $|\lambda| = 1$, then $\overline{\lambda} = \lambda^{-1}$. Thus,

$$U_\lambda^*(a_{i,j})U_\lambda = (\overline{\lambda^i} a_{i,j} \lambda^j) = (\lambda^{-i}\lambda^j a_{i,j}) = (\lambda^{j-i} a_{i,j}).$$

(ii) Let $A = (a_{i,j})$ and $\omega = e^{2\pi i/n}$, which is an n-th root of unity. Note that $\omega, \ldots, \omega^{n-1}$ are precisely the non-trivial n-th roots of unity. Note that

$$\sum_{k=0}^{n-1} \omega^{kl} = \begin{cases} n & \text{if } l = 0 \\ \dfrac{1 - e^{2\pi l i}}{1 - e^{2\pi l i/n}} = 0 & \text{if } 1 \leq l \leq n-1 \end{cases}.$$

We claim that $\sum_{k=1}^{n-1} U_{\omega^k}^* A U_{\omega^k} = n\mathrm{Diag}(A) - A$. Noting that U_1 is the identity matrix, we have,

$$\begin{aligned} A + \sum_{k=1}^{n-1} U_{\omega^k}^* A U_{\omega^k} &= \sum_{k=0}^{n-1} U_{\omega^k}^* A U_{\omega^k} = \sum_{k=0}^{n-1} (\omega^{k(j-i)} a_{i,j}) \\ &= (\sum_{k=0}^{n-1} \omega^{k(j-i)} a_{i,j}) = \mathrm{diag}(a_{1,1}, \ldots, a_{n,n}), \end{aligned}$$

which proves our claim. Since any map of the form $A \mapsto X^* A X$ is completley positive and sums of completely positive maps are completely positive we see that $A \mapsto n\mathrm{Diag}(A) - A$ is completely positive.

(iii) Consider the matrix A which has 1 in every entry. Note that A is positive and that,

$$(n-1)\mathrm{Diag}(A) - A = \begin{bmatrix} n-2 & -1 & \cdots & -1 \\ -1 & n-2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-2 \end{bmatrix}.$$

Now,

$$\left\langle \begin{bmatrix} n-2 & -1 & \cdots & -1 \\ -1 & n-2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right\rangle = -n,$$

which shows that $(n-1)\mathrm{Diag}(A) - A$ is not positive, and hence is not completely positive.

3.7 Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras with unit and let $\varphi_1, \varphi_2 : \mathcal{A} \to \mathcal{B}$ be bounded linear maps with $\varphi_1 \pm \varphi_2$ completely positive. Prove that $\|\varphi_2\|_{cb} \leq \|\varphi_1(\mathbf{1})\|$.

Solution: Define $\varphi_+ = \varphi_1 + \varphi_2$ and $\varphi_- = \varphi_1 - \varphi_2$. Note that $\varphi_1 = \frac{1}{2}(\varphi_+ + \varphi_-)$ and so φ_1 is completely positive. Let $A \in M_n(\mathcal{A})$ with $\|A\| \leq$ and note (by Lemma 3.1.i) that

$$\begin{bmatrix} \mathbf{1} & A \\ A^* & \mathbf{1} \end{bmatrix} \text{ and } \begin{bmatrix} \mathbf{1} & -A \\ -A^* & \mathbf{1} \end{bmatrix},$$

are positive. Since φ_+ and φ_- are completely positive we have,

$$0 \leq (\varphi_+)_{2n}\left(\begin{bmatrix} \mathbf{1} & A \\ A^* & \mathbf{1} \end{bmatrix}\right) + (\varphi_-)_{2n}\begin{bmatrix} \mathbf{1} & -A \\ -A^* & \mathbf{1} \end{bmatrix} = 2\begin{bmatrix} (\varphi_1)_n(\mathbf{1}) & (\varphi_2)_n(A) \\ (\varphi_2)_n(A)^* & (\varphi_1)_n(\mathbf{1}) \end{bmatrix}.$$

Applying the result of exercise 3.2.iv we get $\|(\varphi_2)_n(A)\| \leq \|(\varphi_1)_n(\mathbf{1})\| = \|\varphi_1(\mathbf{1})\|$ since φ_1 is completely positive. Therefore $\|\varphi_2\|_{cb} \leq \|\varphi_1(\mathbf{1})\|$

3.8 Let $\mathcal{A}$ be a C^*-algebra with unit. Define $T_1, T_2 : M_n(\mathcal{A}) \to M_n(\mathcal{A})$ by $T_1((a_{i,j})) = (b_{i,j})$, where $b_{i,i} = \sum_{l=1}^n a_{l,l}$, $b_{i,j} = 0$, for $i \neq j$ and $T_2((a_{i,j})) = (c_{i,j})$ where $c_{i,j} = a_{j,i}$. Fix k and l, $k \neq l$, and define $U_{k,l}^{\pm}$ to be $\mathbf{1}$ in the (k,l)-entry, $\pm\mathbf{1}$ in the (l,k)-entry and 0 elsewhere.

(i) Show that

$$T_1(A) - T_2(A) = \frac{1}{2}\sum_{k \neq l} U_{k,l}^{-*} A U_{k,l}^{-}.$$

(ii) Show that

$$T_1(A) + T_2(A) = \frac{1}{2}\sum_{k \neq l} U_{k,l}^{+*} A U_{k,l}^{+} + \mathrm{Diag}(A).$$

(iii) Deduce that $T_1 \pm T_2$ are completely positive and that $\|T_2\|_{cb} \leq n$.

(iv) By considering, $\mathcal{A} = \mathbb{C}$, show that $\|T_2\|_{cb} = n$.

Solution:

(i) Consider first the case $k = 1$ and note that

$$U_{1,l}^{-*} A U_{1,l}^{-} = \begin{bmatrix} a_{l,l} & \cdots & -a_{l,1} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ -a_{1,l} & \cdots & -a_{1,1} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix},$$

where $a_{l,l}$ is in the $(1,1)$-entry, and $a_{1,1}$ is in the (l,l)-entry. Therefore,

$$\sum_{l \neq 1} U_{1,l}^{-*} A U_{1,l}^{-} = \begin{bmatrix} a_{2,2} + \ldots + a_{n,n} & -a_{2,1} & \cdots & -a_{n,1} \\ -a_{1,2} & a_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1,n} & 0 & \cdots & a_{1,1} \end{bmatrix},$$

In general we have,

$$\sum_{l \neq k} U_{k,l}^{-*} A U_{k,l}^{-} = \begin{bmatrix} a_{k,k} & \cdots & -a_{k,1} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ -a_{1,k} & \cdots & \sum_{l \neq k} a_{l,l} & \cdots & -a_{n,k} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & -a_{k,n} & \cdots & a_{k,k} \end{bmatrix}.$$

Note that for $i \neq j$ the element $-a_{j,i}$ appears twice in the (i,j)-th place, once when $l = j$ and $k = i$ and once when $l = i$ and $k = j$. In the k-th diagonal entry we get $2\sum_{l \neq k} a_{l,l}$. Thus,

$$\begin{aligned} \sum_{k=1}^{n} \sum_{l \neq k} U_{k,l}^{-*} A U_{k,l}^{-} &= \begin{bmatrix} 2\sum_{l \neq 1} a_{l,l} & -2a_{2,1} & \cdots & -2a_{n,1} \\ -2a_{1,2} & 2\sum_{l \neq 2} a_{l,l} & \cdots & -2a_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ -2a_{1,n} & -2a_{2,n} & \cdots & 2\sum_{l \neq n} a_{l,l} \end{bmatrix} \\ &= 2(T_1(A) - T_2(A)) \end{aligned}$$

(ii) Using similar ideas to the those used in the previous part we have that,

$$\begin{aligned} \sum_{k=1}^{n} \sum_{l \neq k} U_{k,l}^{+*} A U_{k,l}^{+} &= \begin{bmatrix} 2\sum_{l \neq 1} a_{l,l} & 2a_{2,1} & \cdots & 2a_{n,1} \\ 2a_{1,2} & 2\sum_{l \neq 2} a_{l,l} & \cdots & 2a_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ 2a_{1,n} & 2a_{2,n} & \cdots & 2\sum_{l \neq n} a_{l,l} \end{bmatrix} \\ &= 2(T_1(A) + T_2(A) - \mathrm{Diag}(A)) \end{aligned}$$

(iii) Using the fact that maps of the type $A \to X^* A X$ and $A \to \mathrm{Diag}(A)$ are completely positive and that sums of completely positive maps are completely positive we see that $T_1 \pm T_2$ are completely

positive. From exercise 3.7 we get that

$$\|T_2\|_{cb} \leq \|T_1(\mathbf{1})\| = \left\| \begin{bmatrix} n\mathbf{1} & 0 & \cdots & 0 \\ 0 & n\mathbf{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n\mathbf{1} \end{bmatrix} \right\| = n$$

(iv) Let $A = (E_{j,i})_{i,j=1}^n \in M_n(M_n)$, where $E_{i,j}$ are the matrix units in M_n. We proved in exercise 1.7 that A was unitary and that $\frac{1}{n}(E_{i,j})_{i,j=1}^n = \frac{1}{n}(T_2)_n(A)$ was a rank one projection. Thus, $\|(T_2)_n\| \geq \|(T_2)_n(A)\| = n$ which proves that $\|T_2\|_{cb} \geq n$. Combining this with the estimate from the previous part of this exercise we get $\|T_2\|_{cb} = n$.

3.9 Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{A}^{op}$ denote the set $\mathcal{A}$ with the same norm and $*$-operation, but with a multiplication defined by $a \circ b = ba$.

(i) Prove that $\mathcal{A}^{op}$ is a C^*-algebra.

(ii) Prove that M_2 and M_2^{op} are $*$-isomorphic via the transpose map.

(iii) Show that the identity map from $\mathcal{A}$ to $\mathcal{A}^{op}$ is always positive.

(iv) Prove that the identity map from M_2 to M_2^{op} is not 2-positive.

(v) (Walter) Let U, V, and X be elements of $\mathcal{A}$ with U, V unitary. Prove that

$$\begin{bmatrix} \mathbf{1} & U & X \\ U^* & \mathbf{I} & V \\ X^* & V^* & \mathbf{1} \end{bmatrix} \geq 0 \tag{8}$$

if and only if $X = UV$.

(vi) Prove that the identity map from $\mathcal{A}$ to $\mathcal{A}^{op}$ is completely positive if and only if $\mathcal{A}$ is commutative.

Solution:

i) It is straightforward to check that $\mathcal{A}$ is a $*$-algebra. The C^*-identity follows from

$$\|a^* \circ a\| = \|aa^*\| = \|(a^*)^* a^*\| = \|a^*\|^2 = \|a\|^2 .$$

ii) It is clear that $\pi : M_2 \to M_2^{op}$ defined by $\pi(A) = A^T$ is a linear map that preserves the $*$-operation. We check that this map is a homorphism:

$$\pi(AB) = (AB)^T = B^T A^T = A^T \circ B^T = \pi(A) \circ \pi(B).$$

iii) Suppose that A is positive in $\mathcal{A}$, then $A = B^*B$ for some B. Thus, $A = B \circ B^* = (B^*)^* \circ B^*$ which shows that A is positive in $\mathcal{A}^{op}$, and so the identity map is positive.

iv) Note that the map $\pi : M_2^{op} \to M_2$ is a $*$-isomorphism and is therefore completely positive. If the identity map $\mathrm{id} : M_2 \to M_2^{op}$ were 2-positive, then $\mathrm{id} \circ \pi : M_2 \to M_2$ would be 2-positive. Note that the matrix

$$A = \begin{bmatrix} E_{1,1} & E_{1,2} \\ E_{2,1} & E_{2,2} \end{bmatrix}$$

is positive in $M_2(M_2)$, but that

$$\mathrm{id} \circ \pi(A) = \begin{bmatrix} E_{1,1} & E_{2,1} \\ E_{1,2} & E_{2,2} \end{bmatrix},$$

is not positive. This contradiction shows that id is not 2-positive.

(v) We present 2 proofs of this result.

Proof 1. Note that an element a of a C^*-algebra $\mathcal{A}$ is positive if and only if x^*ax is positive for all $x \in \mathcal{A}$ Using this fact we get,

$$\begin{aligned}
\begin{bmatrix} \mathbf{1} & U & X \\ U^* & \mathbf{1} & V \\ X^* & V^* & \mathbf{1} \end{bmatrix} \geq 0 \iff & \begin{bmatrix} U^* & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & U & X \\ U^* & \mathbf{1} & V \\ X^* & V^* & \mathbf{1} \end{bmatrix} \begin{bmatrix} U & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix} \geq 0 \\
\iff & \begin{bmatrix} \mathbf{1} & \mathbf{1} & U^*X \\ \mathbf{1} & \mathbf{1} & V \\ X^*U & V^* & \mathbf{1} \end{bmatrix} \geq 0 \\
\iff & \begin{bmatrix} \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & -\mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{1} & U^*X \\ \mathbf{1} & \mathbf{1} & V \\ X^*U & V^* & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & -\mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix} \geq 0 \\
\iff & \begin{bmatrix} 4\mathbf{1} & 0 & U^*X + V \\ 0 & 0 & U^*X - V \\ X^*U + V^* & X^*U - V^* & \mathbf{1} \end{bmatrix} \geq 0
\end{aligned}$$

It follows that the lower-left 2×2 corner of this matrix must also be positive, and so

$$\begin{bmatrix} 0 & U^*X - V \\ X^*U - V & \mathbf{1} \end{bmatrix} \geq 0,$$

which implies by exercise 3.2.i that $U^*X - V = 0$ or $X = UV$, since U is invertible.

Proof 2. **Cholesky lemma**: *Suppose that $P \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $A \in B(\mathcal{K}, \mathcal{H})$ where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. If P is positive and invertible, then the operator matrix*

$$\begin{bmatrix} P & A \\ A^* & B \end{bmatrix} \tag{9}$$

is positive if and only if $B - A^*P^{-1}A$ is positive.

Proof of the lemma: Assume that B is invertible .The matrix in (9) is positive if and only if

$$\begin{bmatrix} P^{-1/2} & 0 \\ 0 & B^{-1/2} \end{bmatrix} \begin{bmatrix} P & A \\ A^* & B \end{bmatrix} \begin{bmatrix} P^{-1/2} & 0 \\ 0 & B^{-1/2} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & P^{-1/2}AB^{-1/2} \\ B^{-1/2}A^*P^{-1/2} & \mathbf{1} \end{bmatrix} \geq 0 \tag{10}$$

which by Lemma 3.1.i happens if and only if

$$B^{-1/2}A^*P^{-1/2}P^{-1/2}AB^{-1/2} \leq \mathbf{1},$$

which is equivalent to

$$A^*P^{-1}A \leq B.$$

In the case where B is not invertible we consider the invertible operator $B + \varepsilon\mathbf{1}$, with $\varepsilon > 0$, and note that the matrix

$$\begin{bmatrix} P & A \\ A^* & B + \varepsilon\mathbf{1} \end{bmatrix}$$

is positive if and only if $B + \varepsilon\mathbf{1} - A^*P^{-1}A \geq 0$. We now have our result by letting $\varepsilon \to 0$.

If we apply this result to the matrix in (8) with $P = \mathbf{1}$, $B = \begin{bmatrix} \mathbf{1} & V \\ V^* & \mathbf{1} \end{bmatrix}$ $A = \begin{bmatrix} U & X \end{bmatrix}$ we get,

$$\begin{aligned} 0 &\leq \begin{bmatrix} \mathbf{1} & V \\ V^* & \mathbf{1} \end{bmatrix} - \begin{bmatrix} U^* \\ X^* \end{bmatrix} \begin{bmatrix} U & X \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{1} - U^*U & V - U^*X \\ V^* - X^*U & \mathbf{1} - X^*X \end{bmatrix} \\ &= \begin{bmatrix} 0 & V - U^*X \\ V^* - X^*U & \mathbf{1} - X^*X \end{bmatrix}. \end{aligned}$$

Once again by exercise 3.2.i we have $V - U^*X = 0$ or $X = UV$.

(vi) Note that if $\mathcal{A}$ is commutative then $\mathcal{A}^{op} = \mathcal{A}$ and the identity map from a C^*-algebra to itself is completely positive.

For the converse we use the fact that the unitary elements span a C^*-algebra. Let U, V be two unitary elements in $\mathcal{A}$. If the identity map is completely positive then

$$\begin{bmatrix} \mathbf{1} & U & UV \\ U^* & \mathbf{1} & V \\ V^*U^* & V^* & \mathbf{1} \end{bmatrix}$$

is positive in $\mathcal{A}^{op}$ and so $UV = U \circ V = VU$, by exercise 3.9.v. Thus, any two unitaries in $\mathcal{A}$ commute and so $\mathcal{A}$ is commutative.