Completely Bounded Maps and Operator Algebras
Chapter 4: Dilation Theorems
Solutions to Exercises

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4.1 Use Stinespring's representation theorem to prove that $\|V\|^2 = \|\varphi\|_{cb}$ when φ is completely positive. Also, use the representation theorem to prove that $\varphi(a)^*\varphi(a) \leq \|\varphi(\mathbf{1})\|\,\varphi(a^*a)$

Solution: We first note by proposition 3.6 that $\|\varphi(\mathbf{1})\| = \|\varphi\| = \|\varphi\|_{cb}$. By Stinespring's theorem we have $\varphi(\mathbf{1}) = V^*V$ and so $\|\varphi(\mathbf{1})\| = \|V^*V\| = \|V\|^2$. We also have,

$$\begin{aligned}
\varphi(a)^*\varphi(a) &= (V^*\pi(a)V)^*V^*\pi(a)V = V^*\pi(a)^*VV^*\pi(a)V \\
&\leq \|V\|^2\, V^*\pi(a)^*\pi(a)V = \|V\|^2\, V^*\pi(a^*a)V \\
&= \|\varphi(\mathbf{1})\|\,\varphi(a^*a).
\end{aligned}$$

4.2 (Multiplicative Domains) In this exercise, we provide an alternative proof of theorem 3.19. Let $\mathcal{A}$ be a C^*-algebra with unit and let $\varphi : \mathcal{A} \to B(\mathcal{H})$ be completely positive, $\varphi(\mathbf{1}) = \mathbf{1}$, with minimal Stinespring representation, $(\pi, V, \mathcal{K})$.

(i) Prove that $\varphi(a)^*\varphi(a) = \varphi(a^*a)$ if and only if $V\mathcal{H}$ is an invariant subspace for $\pi(a)$.

(ii) Use this to give an alternative proof that $\{a \in \mathcal{A} : \varphi(a)^*\varphi(a) = \varphi(a^*a)\} = \{a \in \mathcal{A} : \varphi(ba) = \varphi(b)\varphi(a) \text{ for all } b \in \mathcal{A}\}$. Recall that this set is the *right multiplicative domain* of φ.

(iii) Similarly show that $\varphi(a)^*\varphi(a) = \varphi(a^*a)$ and $\varphi(a)\varphi(a)^* = \varphi(aa^*)$ if and only if $V\mathcal{H}$ is a reducing subspace for $\pi(a)$. Deduce that the set of such elements is a C^*-subalgebra of $\mathcal{A}$. Recall that this subalgebra is the *multiplicative domain* of φ.

Solution: Note that since φ is unital, V is an isometry and $V\mathcal{H}$ is a closed subspace of $\mathcal{K}$. The fact that $\varphi(a) = V^*\pi(a)V$ shows us that, relative to the decomposition $\mathcal{K} = V\mathcal{H} \oplus (V\mathcal{H})^\perp$, $\pi(a)$ is the operator matrix

$$\pi(a) = \begin{bmatrix} \varphi(a) & A(a) \\ B(a) & C(a) \end{bmatrix}.$$

(i) Observe that the invariance of $V\mathcal{H}$ under $\pi(a)$ is equivalent to the requirement that $B(a) = 0$. As π is a $*$-homomorphism we know that $\pi(a^*a) = \pi(a)^*\pi(a)$ or in terms of the $(1,1)$ entry of the corresponding operator matrices

$$\varphi(a^*a) = \varphi(a)^*\varphi(a) + B(a)^*B(a).$$

It is clear from this that $\varphi(a^*a) = \varphi(a)^*\varphi(a)$ if and only if $B(a)^*B(a) = 0$ if and only if $B(a) = 0$.

(ii) If $\varphi(ba) = \varphi(b)\varphi(a)$ for all $b \in \mathcal{A}$ then by setting $b = a^*$ and noting that φ is self-adjoint we see that $\varphi(a^*a) = \varphi(a)^*\varphi(a)$. Conversely assume that $\varphi(a^*a) = \varphi(a)^*\varphi(a)$ and that $b \in \mathcal{A}$. By equating the $(1,1)$ entries of $\pi(ba)$ and $\pi(b)\pi(a)$ we get,

$$\varphi(ba) = \varphi(b)(\varphi(a) + A(b)B(a) = \varphi(b)\varphi(a).$$

(iii) From part 4.2.(i). we have that $\varphi(a)^*\varphi(a) = \varphi(a^*a)$ and $\varphi(a)\varphi(a)^* = \varphi(aa^*)$ if and only if $V\mathcal{H}$ is invariant under both $\pi(a)$ and $\pi(a^*)$ if and only if $V\mathcal{H}$ is reducing for $\pi(a)$.

Note that a is in the multiplicative domain of φ if and only if $V\mathcal{H}$ is reducing which happens if and only if the $(2,1)$ entry of both $\pi(a)$ and $\pi(a^*)$ are 0. This is the collection of $\pi(a)$ that are diagonal, which is a C^*-algebra. Since the multiplicative domain is the inverse image of this set under π we see that it is a C^*-subalgebra of $\mathcal{A}$.

4.3 (Bimodule Maps) Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be C^*-algebras with unit and suppose that $\mathcal{C}$ is contained in both $\mathcal{A}$ and $\mathcal{B}$ with $\mathbf{1}_{\mathcal{C}} = \mathbf{1}_{\mathcal{A}}$ and $\mathbf{1}_{\mathcal{C}} = \mathbf{1}_{\mathcal{B}}$. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is called a $\mathcal{C}$-bimodule map if $\varphi(c_1ac_2) = c_1\varphi(a)c_2$ for all c_1, c_2 in $\mathcal{C}$. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be completely positive.

(i) If $\varphi(\mathbf{1}) = \mathbf{1}$, the prove that φ is a $\mathcal{C}$-bimodule map if and only if $\varphi(c) = c$ for all c in $\mathcal{C}$.

(ii) Prove, in general, that φ is a $\mathcal{C}$-bimodule map if and only if $\varphi(c) = c\varphi(\mathbf{1})$ for all c in $\mathcal{C}$. Moreover, in this case, $\varphi(\mathbf{1})$ commutes with $\mathcal{C}$.

Solution:

(i) See 4.3.(ii).

(ii) We give a proof of part (ii), and part (i) will follow as a special case. Assume first that φ is a $\mathcal{C}$-bimodule map. We have,

$$\varphi(c) = \varphi(c\mathbf{1}^2) = c\varphi(\mathbf{1})\mathbf{1} = c\varphi(\mathbf{1}).$$

Similarly $\varphi(c) = \varphi(\mathbf{1})c$.

For the converse assume $\varphi(c) = c\varphi(\mathbf{1})$ for all $c \in \mathcal{C}$. Note that by taking adjoints we get $\varphi(c^*) = \varphi(\mathbf{1})c^*$. Since $\mathcal{C}$ is self-adjoint it follows that $\varphi(c) = \varphi(\mathbf{1})c$ for all $c \in \mathcal{C}$. We assume that $\mathcal{B} = B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. Let π be a Stinespring dilation of φ on a Hilbert space $\mathcal{K}$ and let $V : \mathcal{H} \to \mathcal{K}$ be the associated linear operator. We will adopt the notation used in Theorem 4.1 (Stinespring's Dilation Theorem). We note that $\mathcal{C}$ is a C^*-subalgebra of $B(\mathcal{H})$ and begin by proving that $Vc = \pi(c)V$. Let $h \in \mathcal{H}$ and note that $Vc(h) = \mathbf{1} \otimes c(h) + \mathcal{N}$ and $\pi(c)V(h) = \pi(c)[\mathbf{1} \otimes h + \mathcal{N}] = c \otimes h + \mathcal{N}$. Thus equality will follow if we show that $\mathbf{1} \otimes c(h) - c \otimes h \in \mathcal{N}$. We have,

$$\begin{aligned}
\langle \mathbf{1} \otimes c(h) - c \otimes h, \mathbf{1} \otimes c(h) - c \otimes h\rangle &= \langle \mathbf{1} \otimes c(h), \mathbf{1} \otimes c(h)\rangle + \langle c \otimes h, c \otimes h\rangle && (1)\\
&\quad - \langle c \otimes h, \mathbf{1} \otimes ch\rangle - \langle \mathbf{1} \otimes c(h), c \otimes h\rangle && (2)\\
&= \langle \varphi(\mathbf{1})c(h), c(h)\rangle + \langle \varphi(c^*c)h, h\rangle && (3)\\
&\quad - \langle \varphi(c)h, c(h)\rangle - \langle \varphi(c^*)c(h), h\rangle && (4)\\
&= \langle \varphi(\mathbf{1})c(h), c(h)\rangle + \langle \varphi(\mathbf{1})c^*c(h), h\rangle && (5)\\
&\quad - \langle \varphi(\mathbf{1})c(h), c(h)\rangle - \langle \varphi(\mathbf{1})c^*c(h), h\rangle = 0 && (6)
\end{aligned}$$

Note that by taking adjoints we have $cV^* = V^*\pi(c)$ for all $c \in \mathcal{C}$. Now,

$$\begin{aligned}
\varphi(c_1ac_2) &= V^*\pi(c_1ac_2)V = V^*\pi(c_1)\pi(a)\pi(c_2)V && (7)\\
&= c_1V^*\pi(a)Vc_2 = c_1\varphi(a)c_2. && (8)
\end{aligned}$$

4.4 Let $\mathcal{D}_n$ be the C^*-subalgebra of diagonal matrices in M_n. Prove that a linear map $\varphi : M_n \to M_n$ is a $\mathcal{D}_n$-bimodule map if and only if φ is the Schur product map, S_T, for some matrix T.

Solution: Let J be the matrix all of whose entries are 1 and let $T = (t_{i,j}) = \varphi(J)$. Let $\{E_{i,j}\}_{i,j=1}^n$ be the set of matrix units for M_n. We have,

$$\varphi(E_{i,j}) = \varphi(E_{j,j} J E_{i,i}) = E_{j,j}\varphi(J)E_{i,i} = t_{i,j}E_{i,j} = T \circ E_{i,j}.$$

By linearity φ must be the Schur product map S_T.