Completely Bounded Maps and Operator Algebras
Chapter 5: Commuting Contractions
Solutions to Exercises

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5.1 Prove that if $(V_1, \mathcal{K}_1)$ and $(V_2, \mathcal{K}_2)$ are two minimal isometric dilations of a contraction operator T on $\mathcal{H}$, then there exists a unitary $U : \mathcal{K}_1 \to \mathcal{K}_2$ such that $Uh = h$ for all $h \in \mathcal{H}$ and $UV_1U^* = V_2$.

Solution: Since $(V_j, \mathcal{K}_j)$, $j = 1, 2$ are minimal isometric dilations we know that

$$\mathcal{H}_j = \{V_j^n h \,:\, h \in \mathcal{H}, n \geq 0\} \tag{1}$$

is dense in $\mathcal{K}_j$. Define a map $U : \mathcal{H}_1 \to \mathcal{H}_2$ by

$$U\left(\sum_{k=-N}^{N} V_1^k h_k\right) = \sum_{k=-N}^{N} V_2^k h_k. \tag{2}$$

We claim that U is an isometry. It follows from this that U has a unique extension to an isometry from $\mathcal{K}_1 \to \mathcal{K}_2$. We will denote this extension by U. As the range of U is dense in $\mathcal{K}_2$, U is unitary. Since V_j is a dilation of T on $\mathcal{K}_j$,

$$V_j^n = \begin{bmatrix} T^n & * \\ * & * \end{bmatrix}, \tag{3}$$

relative to the decomposition $\mathcal{K}_j = \mathcal{H} \oplus \mathcal{H}^\perp$. It follows that if $h, k \in \mathcal{H}$ then $\left\langle V_j^n h, k \right\rangle = \langle T^n h, k \rangle$. We have,

$$\begin{aligned}
\left\| U\left(\sum_{k=-N}^{N} V_1^k h_k\right)\right\|^2 &= \sum_{k,l=-N}^{N} \left\langle V_2^k h_k, V_2^l h_l \right\rangle && (4)\\
&= \sum_{k<l} \left\langle V_2^k h_k, V_2^l h_l \right\rangle + \sum_{k\geq l} \left\langle V_2^k h_k, V_2^l h_l \right\rangle && (5)\\
&= \sum_{k<l} \left\langle h_k, V_2^{l-k} h_l \right\rangle + \sum_{k\geq l} \left\langle V_2^{k-l} h_k, h_l \right\rangle && (6)\\
&= \sum_{k<l} \left\langle h_k, T^{l-k} h_l \right\rangle + \sum_{k\geq l} \left\langle T^{k-l} h_k, h_l \right\rangle && (7)\\
&= \sum_{k<l} \left\langle h_k, V_1^{l-k} h_l \right\rangle + \sum_{k\geq l} \left\langle V_1^{k-l} h_k, h_l \right\rangle && (8)\\
&= \left\| \left(\sum_{k=-N}^{N} V_1^k h_k\right)\right\|^2. && (9)
\end{aligned}$$

Now,

$$
\begin{aligned}
UV_1\left(\sum_{k=-N}^{N} V_1^k h_k\right) &= \left(\sum_{k=-N}^{N} V_2^{k+1} h_k\right) && (10)\\
&= V_2\left(\sum_{k=-N}^{N} V_2^k h_k\right) && (11)\\
&= V_2 U\left(\sum_{k=-N}^{N} V_1^k h_k\right). && (12)
\end{aligned}
$$

Therefore $UV_1 = V_2U$ on the dense subspace $\mathcal{H}_j$. It follows by a limit argument that $UV_1 = V_2U$ or $UV_1U^* = V_2$.

5.2 Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, let $A_n \in B(\mathcal{H}_1, \mathcal{H}_2)$ be a sequence of operators and let $A = (A_{i+j}) : \ell^2(\mathcal{H}_1) \to \ell^2(\mathcal{H}_2)$ be the corresponding Hankel operator. Prove the analogue of Nehari-Page in this setting.

Solution: Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $A = (A_{i+j})$ and let

$$\tilde{A}_n = \begin{bmatrix} 0 & 0 \\ A_n & 0 \end{bmatrix} \in B(\mathcal{H}). \tag{13}$$

By the Nehari-Page theorem the Hankel matrix $\tilde{A} = (\tilde{A}_{i+j})$ is bounded if and only if there exists

$$\tilde{A}_n = \begin{bmatrix} * & * \\ A_n & * \end{bmatrix} \tag{14}$$

for $n < 0$ such that $\|\tilde{B}\|_\infty = \sup_{0<r<1} \|\tilde{B}_r\|_\infty < \infty$ where $\tilde{B}_r(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \tilde{A}_n r^{|n|} e^{in\theta}$. Further $\|(\tilde{A}_{i+j})\| = \|\tilde{B}\|_\infty$. Let $B = \sum_{n=-\infty}^{\infty} A_n e^{in\theta}$. Since B_r is a compression of $\tilde{B}_r$ we have that

$$\|B\| = \sup_{0<r<1} \|B_r\| \le \sup_{0<r<1} \|\tilde{B}_r\| = \|\tilde{B}\| < \infty. \tag{15}$$

All that remains to be shown is that for this choice of A_n, $\|(A_{i+j})\| = \|B\|_\infty$. We have already proven that $\|B\|_\infty \le \|\tilde{B}\|_\infty$. We can identify the Hilbert spaces $\ell^2(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and $\ell^2(\mathcal{H}_1) \oplus \ell^2(\mathcal{H}_2)$ by the map $(h_n^{(1)} \oplus h_n^{(2)}) \mapsto (h_n^{(1)}) \oplus (h_n^{(2)})$. The operator $\tilde{A}$ is identified with the operator $\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}$ and so $\|\tilde{A}\| = \|A\|$. Since A is a compression of the multiplication operator M_B we always have $\|(A_{i+j})\| \le \|M_B\| = \|B\|_\infty$.

5.3 (Caratheodory's Completion Theorem) Let $a_0, \ldots, a_n$ be in $\mathbb{C}$. Use commutant lifting to prove that

$$\left\| \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_n & \cdots & a_1 & a_0 \end{bmatrix} \right\| = \inf \left\| \sum_{j=0}^{n} a_j z^j + \sum_{j=n+1}^{\infty} b_j z^j \right\|_\infty, \tag{16}$$

where the infinum is over sequences $\{b_j\}$ such that the resulting power series is bounded on $\mathbb{D}$ and the ∞-norm is the supremum over $\mathbb{D}$. Moreover, there exists a sequence $\{b_j\}$ where the infimum is attained. Thus, a polynomial can be completed to a power series whose supremum over the disk is bounded by 1 by adding higher order terms if and only if the norm of the corresponding Toeplitz matrix is at most 1.

Deduce that the map

$$f \in H^\infty(\mathbb{T}) \mapsto \begin{bmatrix} \widehat{f}(0) & 0 & \cdots & 0 \\ \widehat{f}(1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \widehat{f}(n) & \cdots & \widehat{f}(1) & \widehat{f}(0) \end{bmatrix} \in M_{n+1},$$

yields an isometric isomorphism of $H^\infty(\mathbb{T})/e^{i(n+1)\theta}H^\infty(\mathbb{T})$ into M_{n+1}. Generalize this to the case where $A_0, \ldots, A_n$ are operators in a (separable) Hilbert Space.

Solution: Denote the matrix on the left side of (16) by R and let

$$S = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \in M_{n+1}.$$

Note first that the minimal unitary dilation of S is the operator U on $L^2(\mathbb{T})$ given by $(Uf)(e^{i\theta}) = e^{i\theta}f(e^{i\theta})$. We have

$$SR = RS = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ a_0 & \ddots & & 0 \\ \vdots & \ddots & \ddots & 0 \\ a_{n-1} & \cdots & a_0 & 0 \end{bmatrix},$$

and so by the commutant lifting theorem there is an operator V commuting with U, with $\|V\| = \|R\|$ such that $RS^m = P_{\mathcal{H}}VU^m|_{\mathcal{H}}$. Since V commutes with U, V is multiplication by $g \in L^\infty$. If $m < 0$, then,

$$\left\langle g, e^{im\theta} \right\rangle = \langle V1, U^m 1\rangle = \left\langle VU^{-m}1, 1\right\rangle = \left\langle P_{\mathcal{H}}VU^{-m}1, 1\right\rangle = \left\langle RS^{-m}1, 1\right\rangle = 0. \tag{17}$$

Therefore, $g(e^{i\theta}) = \sum_{k=0}^\infty a_k e^{ik\theta}$. Let T_g denote the Toeplitz operator associated with g acting on $H^2(\mathbb{T})$. We have,

$$\|R\| = \|V\| = \|T_g\| = \|g\|_{H^\infty(\mathbb{T})} = \|g\|_{H^\infty(\mathbb{D})} = \sup_{z\in\mathbb{D}} \left| \sum_{j=0}^\infty a_j z^j \right|. \tag{18}$$

Let $\{b_j\}$ be a sequence of complex numbers such that $h(z) = \sum_{j=0}^n a_j z^j + \sum_{j=n+1}^\infty b_j z^j$ defines a bounded function on $\mathbb{D}$. Let T_h be the corresponding Toeplitz operator on $H^2(\mathbb{D})$ and note that R

is the compression of T_h to the subspace spanned by $1, z, \ldots, z^n$. Therefore $\|R\| \leq \|T_h\|$. Combining this with (18) we see that,

$$\|R\| = \inf \left\| \sum_{j=0}^{n} a_j z^j + \sum_{j=n+1}^{\infty} b_j z^j \right\|_\infty . \tag{19}$$

Let $f, g \in H^\infty(\mathbb{T})$, let $\Phi : H^\infty(\mathbb{T}) \to M_{n+1}$ be the map

$$f \in H^\infty(\mathbb{T}) \mapsto \begin{bmatrix} \widehat{f}(0) & 0 & \cdots & 0 \\ \widehat{f}(1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \widehat{f}(n) & \cdots & \widehat{f}(1) & \widehat{f}(0) \end{bmatrix} \in M_{n+1}.$$

Since $\widehat{f+g}(k) = \widehat{f}(k) + \widehat{g}(k)$ and $\widehat{\alpha f}(k) = \alpha \widehat{f}(k)$ we see that Φ is linear. To check that Φ is in fact a homomorphism we take note of the fact that $\widehat{fg}(k) = \sum_{j=0}^{k} \widehat{f}(j)\widehat{g}(k-j)$ and compute,

$$\begin{aligned} \Phi(f)\Phi(g) &= \begin{bmatrix} \widehat{f}(0) & 0 & \cdots & 0 \\ \widehat{f}(1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \widehat{f}(n) & \cdots & \widehat{f}(1) & \widehat{f}(0) \end{bmatrix} \begin{bmatrix} \widehat{g}(0) & 0 & \cdots & 0 \\ \widehat{g}(1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \widehat{g}(n) & \cdots & \widehat{g}(1) & \widehat{g}(0) \end{bmatrix} \\ &= \begin{bmatrix} \widehat{f}(0)\widehat{g}(0) & 0 & \cdots & 0 \\ \widehat{f}(1)\widehat{g}(0) + \widehat{f}(0)\widehat{g}(1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \widehat{f}(n)\widehat{g}(0) + \ldots + \widehat{f}(0)\widehat{g}(n) & \cdots & \widehat{f}(1)\widehat{g}(0) + \widehat{f}(0)\widehat{g}(1) & \widehat{f}(0)\widehat{g}(0) \end{bmatrix} \\ &= \Phi(fg) \end{aligned}$$

The kernel of Φ is the set of functions $f \in H^\infty(\mathbb{T})$ such that $\widehat{f}(0) = \ldots = \widehat{f}(n) = 0$, which is precisely $e^{i(n+1)\theta} H^\infty(\mathbb{T})$. Therefore, $H^\infty(\mathbb{T})/e^{i(n+1)\theta} H^\infty(\mathbb{T})$ is isomorphic, via $\tilde{\Phi}(f + e^{i(n+1)\theta} H^\infty(\mathbb{T})) = \Phi(f)$, to a subalgebra of M_{n+1}.

Finally,

$$\begin{aligned} \left\| \tilde{\Phi}(f) \right\| &= \inf \left\{ \|f+g\|_\infty : g \in e^{i(n+1)\theta} H^\infty(\mathbb{T}) \right\} \\ &= \inf \left\{ \left\| \sum_{j=0}^{n} \widehat{f}(j) z^j + \sum_{j=n+1}^{\infty} b_j z^j \right\|_\infty : \sum_{j=n+1}^{\infty} b_j z^j \in H^\infty(\mathbb{D}) \right\} \\ &= \left\| \begin{bmatrix} \widehat{f}(0) & 0 & \cdots & 0 \\ \widehat{f}(1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \widehat{f}(n) & \cdots & \widehat{f}(1) & \widehat{f}(0) \end{bmatrix} \right\| \end{aligned}$$

5.4 Let $\{T_1, \ldots, T_n\}$ be contractions on a Hilbert Space $\mathcal{H}$ (possibly non-commuting). Prove that there exits a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and unitaries $\{U_1, \ldots, U_n\}$ on $\mathcal{K}$ such that

$$T_{i_1}^{k_1} \ldots T_{i_m}^{k_m} = P_{\mathcal{H}} U_{i_1}^{k_1} \ldots U_{i_m}^{k_m}|_{\mathcal{H}},$$

where $m, k_1, \ldots, k_m$ are arbitrary non-negative integers, and $1 \leq i_l \leq n$, for $l = 1, \ldots, n$.

Solution: We begin by recalling the construction used in Bz.-Nagy's dilation theorem. Given a contraction T on a Hilbert space we construct an isometric dilation V on $\ell^2(\mathcal{H})$ by defining

$$V(h_1, h_2, \ldots) = (Th_1, (\mathbf{1} - T^*T)^{1/2}h_1, h_2, h_3, \ldots). \tag{20}$$

We can dilate an isometry V on a Hilbert space $\mathcal{H}$ to a unitary U on $\mathcal{H} \oplus \mathcal{H}$ by

$$U = \begin{bmatrix} V & \mathbf{1} - V^*V \\ 0 & V^* \end{bmatrix}. \tag{21}$$

If we combine these two constructions we see that a contraction T on $\mathcal{H}$ can be dilated to a unitary U on $\ell^2(\mathcal{H}) \oplus \ell^2(\mathcal{H})$.

Given a collection of contractions $T_1, \ldots, T_n$ dilate them as above to $U_1, \ldots, U_n$. Let $h \in \mathcal{H}$ and note that,

$$U_j^k((0, 0, \ldots) \oplus (h, 0, \ldots)) = (0, 0, \ldots) \oplus (T_j^k h, *, *, \ldots). \tag{22}$$

It follows from this that,

$$U_{i_1}^{k_1} \ldots U_{i_m}^{k_m}((0, 0, \ldots) \oplus (h, 0, \ldots)) = (0, 0, \ldots) \oplus (T_{i_1}^{k_1} \ldots T_{i_m}^{k_m} h, *, *, \ldots). \tag{23}$$

Therefore $T_{i_1}^{k_1} \ldots T_{i_m}^{k_m} h = P_{\mathcal{H}} U_{i_1}^{k_1} \ldots U_{i_m}^{k_m} h$.

5.5 (Schaeffer) Let T be a contraction on a Hilbert space $\mathcal{H}$, let $\ell^2_{\mathbb{Z}}(\mathcal{H}) = \sum_{n=-\infty}^{\infty} \oplus \mathcal{H}$ denote the Hilbert space formed as a direct sum of copies of $\mathcal{H}$ indexed by the integers $\mathbb{Z}$. Define an operator matrix $U = (U_{i,j})$ by setting $U_{0,0} = T$, $U_{0,1} = (\mathbf{1} - TT^*)^{1/2}$, $U_{-1,0} = (\mathbf{1} - T^*T)^{1/2}$, $U_{-1,1} = T^*$, $U_{n,n+1} = \mathbf{1}$, for $n \geq 1$ or $n \leq -2$ and $U_{i,j} = 0$ for all other pairs (i, j). Prove that U defines a unitary operator on $\ell^2_{\mathbb{Z}}(\mathcal{H})$ and that if we identify $\mathcal{H}$ with the 0-th copy of $\mathcal{H}$ in $\ell^2_{\mathbb{Z}}(\mathcal{H})$, then $T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}$ for all non-negative integers n.

Solution: We can check by a direct calcuation that all the diagonal entries of UU^* and U^*U are $\mathbf{1}$, and that all other entries, except the $(-1, 0)$ and $(0, -1)$ entries are 0. We find that both the $(-1, 0)$ entry of UU^* and the $(0, -1)$ entry of U^*U are $(\mathbf{1} - T^*T)^{1/2}T^* - T^*(\mathbf{1} - TT^*)^{1/2}$ while both the $(0, -1)$ entry of UU^* and $(-1, 0)$ entry of U^*U are $T(\mathbf{1} - T^*T)^{1/2} - (\mathbf{1} - TT^*)^{1/2}T$.

Thus to prove that U is unitary we need to check that $T(\mathbf{1} - T^*T)^{1/2} = (\mathbf{1} - TT^*)^{1/2}T$. Since T is a contraction $\mathbf{1} - T^*T$ and $\mathbf{1} - TT^*$ are positive operators whose spectrum is contained in the interval $[0, 1]$. Let $f : [0, 1] \to \mathbb{R}$ denote the function $f(t) = \sqrt{1 - t}$. Choose a sequence p_n of polynomials defined on the interval $[0, 1]$ such that $\|p_n - f\|_\infty \to 0$ as $n \to \infty$. Note then that $Tp_n(T^*T) = p_n(TT^*)T$ and so via the continuous functional calculus,

$$T(\mathbf{1} - T^*T)^{1/2} = \lim_{n\to\infty} Tp_n(T^*T) = \lim_{n\to\infty} p_n(TT^*)T = (\mathbf{1} - TT^*)^{1/2}T. \tag{24}$$