Completely Bounded Maps and Operator Algebras
Chapter 6: Completely Positive Maps into M_n
Solutions to Exercises

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6.1 Let $\mathcal{A}$ be any unital C^*-algebra. Give an example of a linear functional, $s : M_n(\mathcal{A}) \to \mathbb{C}$, such that s is unital and positive, but such that the associated linear map , $\varphi_s : \mathcal{A} \to M_n$ has norm n.

Solution: We first consider the case when $\mathcal{A} = \mathbb{C}$ and the linear functional $s : M_n \to \mathbb{C}$ defined by

$$s((a_{i,j})) = \frac{1}{n} \sum_{i,j=1}^{n} a_{i,j}. \tag{1}$$

We have seen in chapter 3 that this functional is positive and unital. We have,

$$\varphi_s(a)_{i,j} = ns(a \otimes E_{i,j}) = a.$$

Thus,

$$\varphi_s(a) = a \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}, \tag{2}$$

which implies,

$$\|\varphi_s(a)\| = a \|J\| = an, \tag{3}$$

and so $\|\varphi_s\| = n$.

For a general C^*-algebra $\mathcal{A}$ choose a state s_0 of this algebra and define a linear functional s by

$$s((a_{i,j})) = \frac{1}{n} \sum_{i,j=1}^{n} s_0(a_{i,j}). \tag{4}$$

Recall that the map $\sigma : M_n(\mathcal{A}) \to \mathcal{A}$ defined by

$$\sigma((a_{i,j})) = \sum_{i,j=1}^{n} a_{i,j}, \tag{5}$$

is completely positive. With this notation $s = \frac{1}{n}(s_0 \circ \sigma)$ which, being the composition of two positive maps, is positive. We have

$$s(\mathbf{1}) = \frac{1}{n} \sum_{i,j=1}^{n} s_0(\delta_{i,j}\mathbf{1}) = 1. \tag{6}$$

$\varphi_s(a)_{i,j} = ns(a \otimes E_{i,j}) = s_0(a)$ and so

$$\varphi_s(a) = \begin{bmatrix} s_0(a) & \cdots & s_0(a) \\ \vdots & \ddots & \vdots \\ s_0(a) & \cdots & s_0(a) \end{bmatrix}. \tag{7}$$

It follows that $\|\varphi_s(a)\| = s_0(a)n$. Since s_0 is a state $\|s_0\| = 1$ and so $\|\varphi_s\| \leq n$. By setting $a = \mathbf{1}$ we get that $\|\varphi_s\| = n$.

6.2 Let $\varphi : \mathcal{S} \to M_n$ be positive and set $\varphi(\mathbf{1}) = P$. Let Q be the projection onto the range of P and let R be positive with $(\mathbf{1} - Q)R = 0$, $RPR = Q$. Let $\psi : \mathcal{S} \to M_n$ be any positive, unital map and set $\varphi'(a) = R\varphi(a)R + (\mathbf{1} - Q)\psi(a)(\mathbf{1} - Q)$.

(i) Show that φ' is a unital, positive map.

(ii) Show that is $(a_{i,j})$ is positive in $M_n(\mathcal{S})$, but $\varphi_k((a_{i,j}))$ is not positive, then $\varphi'_k((a_{i,j}))$ is not positive either.

(iii) Deduce the equivalence of i) and ii) in Theorem 6.6.

Solution:

(i) Suppose that $a \geq 0$. Since φ and ψ are positive, $\varphi(a)$ and $\psi(a)$ are positive. Since R and $(1-Q)$ are self-adjoint $R\varphi(a)R$ and $(1-Q)\psi(a)(1-Q)$ are positive and so $\varphi'(a)$ is positive. We have,

$$\begin{aligned} \varphi'(\mathbf{1}) &= R\varphi(\mathbf{1})R + (1-Q)\psi(\mathbf{1})(1-Q) && (8) \\ &= RPR + (1-Q)^2 = Q + (1-Q) = 1 && (9) \end{aligned}$$

(ii) We claim that $\operatorname{range}\varphi(a) \subseteq \operatorname{range} P$ for all $a \in \mathcal{S}$. Indeed if $A, B \in B(\mathcal{H})$ are two positive operators such that $A \leq B$, then $\ker B \subseteq \ker A$. To see this assume that $Bh = 0$ and note that this implies that $\langle Ah, h\rangle = 0$. Thus,

$$0 = \langle Ah, h\rangle = \left\langle A^{1/2}h, A^{1/2}h\right\rangle = \left\|A^{1/2}h\right\|^2.$$

Hence, $Ah = A^{1/2}(A^{1/2}h) = 0$.

The fact that φ is positive implies that $\varphi(a) \leq \|a\| P$ for $a \geq 0$. It follows that $\ker\varphi(a) \supseteq \ker P$, which in turn implies on taking orthogonal complements that $\operatorname{range}\varphi(a) \subseteq \operatorname{range} P$. Therefore, since φ is linear and every element of $\mathcal{S}$ is a sum of at most 4 positive elements, $\operatorname{range}\varphi(x) \subseteq \operatorname{range} P$ for all $x \in \mathcal{S}$. If $z \in \mathbb{C}^n$, then we may write $z = x + y$ with $x \in \operatorname{range} P$ and $y \in \ker P$. Since $\operatorname{range}\varphi(a) \subseteq \operatorname{range} P$ we have

$$\langle \varphi(a)z, z\rangle = \langle \varphi(a)x, x\rangle. \tag{10}$$

Let $(a_{i,j}) \in M_k(\mathcal{S})$ be positive and assume that $\varphi_k((a_{i,j}))$ is not positive. By our observation in (10) we can choose $x_1, \ldots, x_n \in \operatorname{range} P$ such that,

$$\sum_{i,j=1}^{k} \langle \varphi(a_{i,j})x_j, x_i\rangle \ngeq 0. \tag{11}$$

Set $z_j = PRx_j \in \operatorname{range} P$,

$$\begin{aligned}
\sum_{i,j=1}^{k} \langle \varphi'(a_{i,j})z_j, z_i\rangle &= \sum_{i,j=1}^{k} \langle R\varphi(a_{i,j})Rz_j, z_i\rangle + \sum_{i,j=1}^{k} \langle ((1-Q)\psi(a_{i,j})(1-Q)z_j, z_i\rangle && (12)\\
&= \sum_{i,j=1}^{k} \langle \varphi(a_{i,j})Rz_j, Rz_i\rangle + \sum_{i,j=1}^{k} \langle \psi(a_{i,j})(1-Q)z_j, Qz_i\rangle && (13)\\
&= \sum_{i,j=1}^{k} \langle \varphi(a_{i,j})Qx_j, Qx_i\rangle + \sum_{i,j=1}^{k} \langle \psi(a_{i,j})(1-Q)z_j, Qz_i\rangle && (14)\\
&= \sum_{i,j=1}^{k} \langle \varphi(a_{i,j})x_j, x_i\rangle \ngeq 0. && (15)
\end{aligned}$$

(iii) Clearly (i) implies (ii) in Theorem 6.6. Now assume that every unital, posiitve map $\varphi : \mathcal{S} \to M_n$ is completely positive and that $\psi : \mathcal{S} \to M_n$ is a positive map. By the previous part of this exercise we can construct a unital, positive map ψ' such that ψ is k-positive whenever ψ' is k-positive. By our hypothesis ψ' is completely positive and so ψ is completely positive.

6.3 Assume that the equivalent conditions of Theorem 6.8 are not met. Show that then there always exists a unital, positive map $\varphi : \mathcal{S} \to M_2$ which is not contractive.

Solution: If the equivalent conditons of theorem 6.8 are not met then there exists an element x of norm at most 1 and a positive map φ such that $\|\varphi(x)\| > \|\varphi(\mathbf{1})\|$. It follows that

$$\begin{bmatrix} \varphi(\mathbf{1}) & \varphi(x) \\ \varphi(x)^* & \varphi(\mathbf{1}) \end{bmatrix},$$

is not positive. Construct a positive unital map φ' from φ as in exercise 6.2.(i) and note that

$$\begin{bmatrix} \varphi'(\mathbf{1}) & \varphi'(x) \\ \varphi'(x)^* & \varphi'(\mathbf{1}) \end{bmatrix},$$

is not positive by exercise 6.2.(ii). Hence, $\|\varphi'(x)\| > \|\varphi'(\mathbf{1})\| = \|\mathbf{1}\| = 1$. Thus φ' is unital and positive but not contractive.

6.5 Let $\mathcal{S}$ be an operator system. Prove that the following are equivalent:

(i) For every C^*-algebra $\mathcal{B}$, every positive $\varphi : \mathcal{S} \to \mathcal{B}$ is n-positive.

(ii) $\mathcal{S}^+ \otimes M_n^+$ is dense in $M_n(\mathcal{S})$.

Solution: (i) implies (ii): Let $\mathcal{B} = M_n$ and note then that any positive map φ from $\mathcal{S}$ into M_n is n-positive. It follows from theorem 6.1 that φ is completely positive and so by Theorem 6.6 $\mathcal{S}^+ \otimes M_n^+$ is dense in $M_n(\mathcal{S})^+$.

(ii) implies (i): We may assume that $\mathcal{B} = B(\mathcal{H})$. By theorem 6.6 we know that any positive map $\varphi : \mathcal{S} \to M_n$ is completely positive. Let $(a_{i,j})$ be positive in $M_n(\mathcal{S})$. We need to check that $\varphi_n((a_{i,j}))$ is positive in $B(\mathcal{H}^{(n)})$. Let $h_1, \ldots, h_n \in \mathcal{H}$ and let $\mathcal{F}$ be the m-dimensional subspace spanned by

$\{h_1, \ldots, h_n\}$. Let $\psi : \mathcal{S} \to B(\mathcal{F})$ be the compression of φ to $\mathcal{F}$. If we identify M_m with $B(\mathcal{F})$ and note that M_m can be identified with the subalgebra of M_n via

$$A \in M_m \mapsto \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \in M_n,$$

then we see that ψ is n-positive by ii). Therefore,

$$\sum_{i,j=1}^{n} \langle \varphi(a_{i,j})h_j, h_i \rangle = \sum_{i,j=1}^{n} \langle \psi(a_{i,j})h_j, h_i \rangle \geq 0.$$

6.6 Use corollary 6.7 to give an alternate proof of the fact that every positive map with domain $C(X)$ is completely positive.

Solution: Let $F \in M_n(C(X))^+$. Since X is compact we may choose a finite cover of open sets $\{U_j\}_{j=1}^m$ for X and sequence of points $x_j \in U_j$ such that $\|F(x) - F(x_j)\| < \varepsilon$ for all $x \in U_j$. Let p_j be a partition of unity subordinate to the cover U_j. Note that if $p_j(x) \neq 0$, then $\|F(x) - F(x_j)\| < \varepsilon$. It follows that,

$$\begin{aligned}
\left\| F(x) - (\sum_{j=1}^{m} p_j \otimes F(x_j))(x) \right\| &= \left\| F(x) - \sum_{j=1}^{m} p_j(x) F(x_j) \right\| && (16) \\
&= \left\| \sum_{j=1}^{m} p_j(x)(F(x) - F(x_j)) \right\| && (17) \\
&\leq \sum_{j=1}^{m} \|p_j(x)(F(x) - F(x_j))\| && (18) \\
&= \sum_{j=1}^{m} p_j(x) \|F(x) - F(x_j)\| < \varepsilon && (19)
\end{aligned}$$

We have shown that $C(X)^+ \otimes M_n^+$ is dense in $M_n(C(X))^+$ and so by corollary 6.7 every positive map with domain $C(X)$ is completely positive.