Completely Bounded Maps and Operator Algebras
Chapter 7: Arveson's Extension Theorems
Solutions to Exercises

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7.1 Let X and Y be Banach spaces and for $f \in X^*$, $g \in Y^*$, define $L_{f,g} \in B(X, Y^*)$ by $L_{f,g}(x) = f(x)g$. By considering these operators prove that the map $j : X \otimes Y \to B(X, Y^*)^*$ defined in this chapter has the following properties:

(i) j is linear,

(ii) $\|j(x \otimes y)\| = \|x\| \, \|y\|$,

(iii) j is one-to-one.

Since j is one-to-one, the identification of $X \otimes Y$ with $j(X \otimes Y)$ endows $X \otimes Y$ with a norm (rather than just a seminorm). Conclude that Z, of Lemma 7.1, can be identified with the completion of $X \otimes Y$ with respect to this norm.

A norm on the tensor product of two normed spaces that satisifes $\|x \otimes y\| = \|x\| \, \|y\|$ is called a cross-norm.

Solution:

(i) We define $j : X \otimes Y \to B(X, Y^*)^*$ as $j(x \otimes y)(L) = L(x)(y)$ and extend this map linearly, which guarantees that j is linear.

(ii) Suppose that $L \in B(X, Y^*)$. We have,

$$|j(x \otimes y)(L)| = |L(x)(y)| \leq \|L(x)\| \, \|y\| \leq \|x\| \, \|y\| \, .$$

It follows from this that $\|j(x \otimes y)\| \leq \|x\| \, \|y\|$. To prove the reverse inequality choose linear functionals $f \in X^*$, $g \in Y^*$ such that $f(x) = \|x\|$, $g(y) = \|y\|$. Note that,

$$j(x \otimes y)(L_{f,g}) = f(x)g(y) = \|x\| \, \|y\| \, ,$$

and so $\|j(x \otimes y\| = \|x\| \, \|y\|$.

(iii) Let $v \in X \otimes Y$ and suppose that $j(v) = 0$. We can write, $v = \sum_{i=1}^m x_i \otimes y_i$ where the set $\{x_1, \ldots, x_m\}$ is linearly independent. Choose linear functionals $f_j \in X^*$ such that $f_j(x_i) = \delta_{i,j}$. Let $g \in Y^*$ and consider,

$$0 = j(v)(L_{f_j,g}) = \sum_{i=1}^m f_j(x_i)g(y_i) = g(y_j).$$

Thus $g(y_j) = 0$ for all $g \in Y^*$ which implies that $y_j = 0$ for all $j = 1, \ldots, m$. Hence $v = 0$ and j is one-to-one.

7.3 Let $\mathcal{A}$ be an operator algebra contained in the C^*-algebra $\mathcal{B}$, let $\rho : \mathcal{A} \to B(\mathcal{H})$ be a completely contractive, unital homomorphism, and let $\pi_i : \mathcal{B} \to B(\mathcal{K}_i), i = 1, 2$, define minimal $\mathcal{B}$-dilations of ρ. Define completely positive maps, $\varphi_i : \mathcal{B} \to B(\mathcal{H})$, by $\varphi_i(b) = P_{\mathcal{H}}\pi_i(b)|_{\mathcal{H}}, i = 1, 2$.

(i) Show that there exists a unitary $U : \mathcal{K}_1 \to \mathcal{K}_2$ with $Uh = h$ for h in $\mathcal{H}$, and $U^*\pi_2(b)U = \pi_1(b)$ if and only if $\varphi_1 = \varphi_2$. Such dilations are called *unitarily equivalent*.

(ii) Show that there is a one-to-one correspondence between unitarily equivalent, minimal $\mathcal{B}$ dilations of ρ and completely positive extensions of ρ to $\mathcal{B}$.

(iii) Show that the set of completely positive extensions of ρ is a compact, convex set in the BW-topology on $CP(\mathcal{B}, \mathcal{H})$.

Solution:

(i) Assume first that such a unitary does exist. Since $Uh = h$ for all $h \in \mathcal{H}$ we see that $U^*h = h$ for all $h \in \mathcal{H}$ and so $P_{\mathcal{H}}U^* = P_{\mathcal{H}}$ We have

$$\begin{aligned} \varphi_1(b)(h) &= P_{\mathcal{H}}\pi_1(b)(h) = P_{\mathcal{H}}U^*\pi_2(b)Uh \\ &= P_{\mathcal{H}}U^*\pi_2(b)h = P_{\mathcal{H}}\pi_2(b)(h) \\ &= \varphi_2(b)(h), \end{aligned}$$

which shows that $\varphi_1 = \varphi_2$.

Assume that $\varphi_1 = \varphi_2 = \varphi$. Define $V_i : \mathcal{H} \to \mathcal{K}_i$ by $V_i h = h$. If we decompose $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$, and $h \in \mathcal{H}, h' \in \mathcal{H}^\perp$, then $V_i^* : \mathcal{K} \to \mathcal{H}$ is given by $V_i^*(h + h') = h$. Therefore $V_i^*\pi_i(b)V_i = \varphi_i(b)$ for any $b \in \mathcal{B}$. Since $\pi(\mathcal{B})\mathcal{H}$ is dense in $\mathcal{K}_i$ it follows that $(\pi_i, \mathcal{K}_i, V_i)$ are minimal Stinespring dilations of φ. Thus there exists a unitary $U : \mathcal{K}_1 \to \mathcal{K}_2$ such that $U^*\pi_2(b)U = \pi_1(b)$.

(ii) Given a minimal $\mathcal{B}$-dilation we define an extension $\varphi : \mathcal{B} \to B(\mathcal{H})$ of ρ by $\varphi(b) = P_{\mathcal{H}}\pi(b)|_{\mathcal{H}}$. The preceeding part of this exercise shows that unitarily equivalent $\mathcal{B}$-dilations give rise to the same completely positive extension.

(iii) Let φ_1, φ_2 be two completely positive extensions of ρ. For $0 \le t \le 1$ the map $\varphi = t\varphi_1 + (1-t)\varphi_2$ is completely positive. If $a \in \mathcal{A}$, then

$$\varphi(a) = t\varphi_1(a) + (1-t)\varphi_2(a) = t\rho(a) + (1-t)\rho(a) = \rho(a). \tag{1}$$

Let $\mathcal{E}$ denote the set of completely positive extensions of ρ and note that this set is a subset of the $CP(\mathcal{B}, \mathcal{H}, 1)$ which is compact in the BW-topology. It is enough to show that $\mathcal{E}$ is closed in the BW-topology. Let $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ be a net in $\mathcal{E}$ and assume that $\varphi_\lambda \to \varphi$. Since $CP(\mathcal{B}, \mathcal{H}, 1)$ is closed φ is completely positive. As φ_λ is completely positive and unital, φ_λ is contractive. From Lemma 7.3 we have that

$$\langle \varphi_\lambda(b)x, y\rangle \to \langle \varphi(b)x, y\rangle, \tag{2}$$

for all $x, y \in \mathcal{H}$ and $b \in \mathcal{B}$. If $a \in \mathcal{A}$, then

$$\langle \rho(a)x, y\rangle = \langle \varphi_\lambda(a)(x), y\rangle \to \langle \varphi(a)x, y\rangle. \tag{3}$$

Therefore $\langle \rho(a)x, y\rangle = \langle \varphi(a)x, y\rangle$ for all $x, y \in \mathcal{H}$. Hence, $\varphi(a) = \rho(a)$ and $\varphi \in \mathcal{E}$.

7.4 (Extension of Bimodule Maps) Let $\mathcal{A}, \mathcal{C}$ be C^*-algebras, let $\mathcal{S}$ be an operator system, and suppose that $\mathcal{C} \subseteq \mathcal{S} \subseteq \mathcal{A}$. If $\mathcal{C} \subseteq B(\mathcal{H})$, then $\varphi : \mathcal{S} \to B(\mathcal{H})$ is $\mathcal{C}$-bimodule map provided $\varphi(c_1 a c_2) = c_1\varphi(a)c_2$. Prove that if $\varphi : \mathcal{S} \to B(\mathcal{H})$ is a completely positive $\mathcal{C}$-bimodule map, then every completely positive extension of φ to $\mathcal{A}$ is also a $\mathcal{C}$-bimodule map.

Solution: To check that a completely positive extension ψ of φ is $\mathcal{C}$-bimodule it is enough by exercise 4.3.ii to check that $\psi(c) = c\psi(\mathbf{1})$. Since ψ is an extension of φ this is the same as saying that $\varphi(c) = c\varphi(\mathbf{1})$. The fact that φ is $\mathcal{C}$-bimodule gives,

$$\varphi(c) = \varphi(c\mathbf{1}^2) = c\varphi(\mathbf{1})\mathbf{1} = c\varphi(\mathbf{1}).$$

7.5 Let $\mathcal{B} \subseteq B(\mathcal{H})$ be a unital C^*-algebra. Prove that $\mathcal{B}$ is injective if and only if there exists a completely positive map $\varphi : B(\mathcal{H}) \to \mathcal{B}$ such that $\varphi(b) = b$ for all b in $\mathcal{B}$. Show that φ is necessarily a $\mathcal{B}$-bimodule map. A map with the above properties is called a *completely positive conditional expectaton.*

Solution: Assume that $\mathcal{B}$ is injective. Note that the identity map $I : \mathcal{B} \to \mathcal{B}$ is completely positive and that $\mathcal{B}$ is an operator system contained in $B(\mathcal{H})$. Since $\mathcal{B}$ is injective I has a completely positive extension $\varphi : B(\mathcal{H}) \to \mathcal{B}$. As φ extends I, $\varphi(b) = I(b) = b$ for all $b \in \mathcal{B}$.

For the converse assume that there is a completely positive map $\varphi : B(\mathcal{H}) \to \mathcal{B}$ such that $\varphi(b) = b$ for all $b \in \mathcal{B}$. Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{S}$ be an operator system contained in $\mathcal{A}$ and suppose that $\psi : \mathcal{S} \to \mathcal{B}$ is completely positive. Let $j : \mathcal{B} \to B(\mathcal{H})$ denote the inclusion map and note that $j \circ \psi : \mathcal{S} \to B(\mathcal{H})$ is completely positive. Since $B(\mathcal{H})$ is injective this map extends to a completely positive map $\theta : \mathcal{A} \to B(\mathcal{H})$. Thus $\varphi \circ \theta : \mathcal{A} \to \mathcal{B}$ is completely positive. We need to check that $\varphi \circ \theta$ extends ψ. For $x \in \mathcal{S}$ we have,

$$\varphi \circ \theta(x) = \varphi(j(\psi(x))) = \varphi(\psi(x)) = \psi(x),$$

since $\varphi(b) = b$ for all $b \in \mathcal{B}$ and $\psi(x) \in \mathcal{B}$.