Completely Bounded Maps and Operator Algebras
Chapter 8: Completely Bounded Maps
Solutions to Exercises

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8.1 Show that $\operatorname{Re}(\varphi_n) = (\operatorname{Re}\varphi)_n$ and that $(\varphi_n)^* = (\varphi^*)_n$.

Solution:Let $(a_{i,j})_{i,j=1}^n \in M_n(\mathcal{M})$. We have,

$$\begin{aligned}(\varphi^*)_n((a_{i,j})) &= (\varphi^*(a_{i,j})) = (\varphi(a_{i,j}^*)^*) = (\varphi(a_{j,i}^*))^* \\ &= \varphi_n((a_{j,i}^*))^* = \varphi_n((a_{i,j})^*)^* \\ &= (\varphi_n)^*((a_{i,j}))\end{aligned}$$

Using this we get,

$$\begin{aligned}2(\operatorname{Re}\varphi_n) &= \varphi_n + (\varphi_n)^* \\ &= \varphi_n + (\varphi^*)_n \\ &= (\varphi + \varphi^*)_n \\ &= 2(\operatorname{Re}\varphi)_n\end{aligned}$$

8.2 Let $\varphi : \mathcal{M} \to \mathcal{B}$, let H and K be in M_n, and let A be in $M_n(\mathcal{M})$. Prove that $\varphi_n(HAK) = H\varphi_n(A)K$. Thus, $\varphi_n : M_n(\mathcal{M}) \to M_n(\mathcal{B})$ is an M_n-bimodule map.

Solution: Let $H = (\lambda_{i,j})$, $K = (\mu_{i,j})$ and $A = (a_{i,j})$. Note that the (i,j) entry of HAK is

$$\sum_{l=1}^n \sum_{k=1}^n \lambda_{i,k} a_{k,l} \mu_{l,j}. \tag{1}$$

It follows that,

$$\begin{aligned}\varphi_n(HAK) &= \varphi_n\left(\left(\sum_{k,l=1}^n \lambda_{i,k} a_{k,l} \mu_{l,j}\right)_{i,j=1}^n\right) &(2)\\ &= \left(\varphi\left(\sum_{k,l=1}^n \lambda_{i,k} a_{k,l} \mu_{l,j}\right)\right)_{i,j=1}^n &(3)\\ &= (\sum_{k,l=1}^n \lambda_{i,k} \varphi(a_{k,l}) \mu_{l,j}) &(4)\\ &= H(\varphi(a_{i,j}))K = H\varphi_n(A)K. &(5)\end{aligned}$$

8.3 Verify the claim of Theorem 8.4.

Solution: Let $\{E_{i,j}\}_{i,j=1}^2$ denote the matrix units in $M_2 \subseteq M_2(\mathcal{A})$. Let $\mathcal{K} = \operatorname{range}\pi_1(E_{1,1})$. Note since π_1 is a $*$-homomorphism that $\pi_1(E_{1,1})$ is a projection and thus $\mathcal{K}$ is closed. We claim that $\mathcal{K}_1 \cong \mathcal{K} \oplus \mathcal{K}$. Define $U : \mathcal{K} \oplus \mathcal{K} \to \mathcal{K}_1$ by,

$$U(x \oplus y) = x + \pi_1(E_{2,1})y.$$

It is straightforward that U is linear. We now prove that it is an isometry.

$$\begin{aligned}
\|U(x \oplus y)\|^2 &= \langle x, x\rangle + \langle \pi_1(E_{2,1})y, \pi_1(E_{2,1})y\rangle + 2\mathrm{Re}\ \langle x, \pi_1(E_{2,1})y\rangle && (6)\\
&= \|x\|^2 + \langle \pi_1(E_{1,2}E_{2,1})y, y\rangle + 2\mathrm{Re}\ \langle \pi_1(E_{1,1})x, \pi_1(E_{2,1})y\rangle && (7)\\
&= \|x\|^2 + \langle \pi_1(E_{1,1})y, y\rangle + 2\mathrm{Re}\ \langle \pi_1(E_{1,2}E_{1,1})x, y\rangle && (8)\\
&= \|x\|^2 + \|y\|^2 = \|x \oplus y\|^2 . && (9)
\end{aligned}$$

Now let $z \in \mathcal{K}_1$ and note that

$$\begin{aligned}
z = I_{\mathcal{K}_1} z &= \pi(E_{1,1})z + \pi(E_{2,2})z && (10)\\
&= \pi(E_{1,1})z + \pi(E_{2,1})\pi(E_{1,1})\pi(E_{1,2})z && (11)\\
&= U(\pi(E_{1,1})z \oplus \pi(E_{1,1})\pi(E_{1,2})z). && (12)
\end{aligned}$$

This shows that U is surjective and is therefore a unitary.

Define $P_j : \mathcal{K} \oplus \mathcal{K} \to \mathcal{K}$ by $P_j(x_1 \oplus x_2) = x_j$ and $\pi : \mathcal{A} \to B(\mathcal{K})$ by

$$\pi(a) = P_1\pi_1(a)P_1^*.$$

Note that $P_i^*P_i = I_{\mathcal{K}}$. It follows that π is a unital, linear map and we check that it is a $*$-homomorphism. Let $a, b \in \mathcal{A}$,

$$\pi(ab) = P_i\pi_1(ab)P_i^* = P_i(\pi_1(a)\pi_1(b))P_i^* = P_i\pi_1(a)P_i^*P_i\pi_1(b)P_i^* = \pi(a)\pi(b). \tag{13}$$

and,

$$\pi(a^*) = P_i\pi_1(a)^*P_i^* = (P_i^*\pi_1(a)P_i)^* = \pi(a)^* \tag{14}$$

To complete the verification we must show that the (i, j)-th entry of $\pi_1((a_{i,j}))$ is $\pi(a_{i,j})$. Note that the (i, j)-th entry is $P_i\pi_1((a_{i,j}))P_j^*$. Now,

$$\begin{aligned}
P_i\pi_1((a_{i,j}))P_j^* &= \sum_{k,l=1}^{2} P_i\pi_1(a_{k,l} \otimes E_{k,l})P_j^*\\
&= \sum_{k,l=1}^{2} P_i\pi_1(a_{k,l} \otimes (E_{k,1}E_{1,1}E_{1,l}))P_j^*\\
&= \sum_{k,l=1}^{2} P_i\pi_1(E_{k,1})\pi_1(a_{k,l} \otimes E_{1,1})\pi_1(E_{1,l})\\
&= P_i\pi_1(E_{i,1})\pi_1(a_{i,j} \otimes E_{1,1})\pi_1(E_{1,j})P_j^*\\
&= P_1\pi_1(a_{i,j}E_{1,1})P_1^* = \pi(a_{i,j}).
\end{aligned}$$

8.4 Show that if φ is completely bounded, and $\varphi(a) = V_1^*\pi(a)V_2$ is the representation of theorem 8.4 with $\|V_1\| = \|V_2\|$, then setting $\varphi_i(a) = V_i^*\pi(a)V_i$, yields the map Φ of Theorem 8.3.

Solution: Note that

$$\|\varphi\|_{cb} = \|V_1\|\,\|V_2\| = \|V_i\|^2 = \|V_i^*V_i\| = \|V_i^*\pi(\mathbf{1})V_i\| \le \|\varphi_i(\mathbf{1})\|\,. \tag{15}$$

Also,

$$\begin{aligned} \|\varphi_i(a)\| &= \|V_i^*\pi(a)V_i\| \leq \|V_i^*\| \, \|\pi(a)\| \, \|V_i\| \leq \|V_i^*\| \, \|V_i\| \, \|a\| && (16) \\ &= \|V_i\|^2 \, \|a\| = \|\varphi\|_{cb} \, \|a\| && (17) \end{aligned}$$

Therefore $\|\varphi\|_{cb} = \|\varphi_i\|_{cb}$. If $\|\varphi\|_{cb} = 1$, then we may choose V_i, $i = 1, 2$, to be isometries. Therefore, $\varphi_i(\mathbf{1}) = V_i\pi(\mathbf{1})V_i = V_i^*V_i = \mathbf{1}$.

We compute,

$$\varphi^*(a) = \varphi(a^*)^* = (V_1^*\pi(a^*)V_2)^* = V_2^*\pi(a)V_1. \tag{18}$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}$. We have,

$$\begin{aligned} \Phi(A) &= \begin{bmatrix} \varphi_1(a) & \varphi(b) \\ \varphi^*(c) & \varphi_2(d) \end{bmatrix} && (19) \\ &= \begin{bmatrix} V_1^*\pi(a)V_1 & V_1^*\pi(b)V_2 \\ V_2^*\pi(c)V_1 & V_2^*\pi(d)V_2 \end{bmatrix} && (20) \\ &= \begin{bmatrix} V_1^* & 0 \\ 0 & V_2^* \end{bmatrix} \begin{bmatrix} \pi(a) & \pi(b) \\ \pi(c) & \pi(d) \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} && (21) \\ &= V^*\pi_2(A)V. && (22) \end{aligned}$$

This shows that Φ is completely positive, since π is a $*$-homomorphism

8.5 Prove that the conclusions of Theorems 8.2, 8.3 and 8.5 still hold when the range is changed from $B(\mathcal{H})$ to an arbitrary injective C^*-algebra.

Solution: Throughout this solution $\mathcal{B}$ will denote an injective C^*-algebra, which is represented on $\mathcal{H}$. By exercise 7.5 there exists a completely positive map $\theta : B(\mathcal{H}) \to \mathcal{B}$ such that $\theta(b) = b$ for all $b \in \mathcal{B}$. *Theorem 8.2* Let $\varphi : \mathcal{M} \to \mathcal{B}$ be completely bounded. Extend φ, by Wittstock's theorem, to a completely bounded map $\psi : \mathcal{A} \to B(\mathcal{H})$. Notice that the map $\theta \circ \psi : \mathcal{A} \to \mathcal{B}$ is completely bounded. Let $a \in \mathcal{M}$, note that $\varphi(a) \in \mathcal{B}$ and so

$$\theta \circ \psi(a) = \theta \circ \varphi(a) = \varphi(a).$$

Thus, $\theta \circ \psi$ is a completely bounded extension of φ. Since θ is unital and completely positive we have $\|\theta\| = \|\theta\|_{cb} = 1$. Using this fact we get,

$$\|\varphi\|_{cb} \leq \|\theta \circ \psi\|_{cb} \leq \|\theta\|_{cb} \, \|\psi\|_{cb} = \|\varphi\|_{cb} \, . \tag{23}$$

Theorem 8.3 Let $\mathcal{A}$ be a C^*-algebra with unit, let $\psi : \mathcal{A} \to \mathcal{B}$ be completely bounded. Then there exists completely positive maps $\psi_i : \mathcal{A} \to \mathcal{B}$ with $\|\psi_i\|_{cb} = \|\psi\|_{cb}$, $i = 1, 2$, such that the map $\Psi : M_2(\mathcal{A}) \to M_2(\mathcal{B})$ given by

$$\Psi \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \psi_1(a) & \psi(b) \\ \psi^*(c) & \psi_2(d) \end{bmatrix} \tag{24}$$

is completely positive. Moreover, if $\|\psi\|_{cb} = 1$, then we may take $\psi_i(1) = \mathbf{1}$, $i = 1, 2$.

We may assume that $\|\psi\|_{cb} = 1$. Let $\theta : B(\mathcal{H}) \to \mathcal{B}$ be a completely positive projection. Construct with ψ is place of φ the maps φ_i, $i = 1, 2$ and Φ as in Theorem 8.3. Let $\psi_i = \theta \circ \varphi_i$. As θ fixes $\mathcal{B}$ we

have that $\theta \circ \psi = \psi$ and $\theta \circ \psi^* = \psi^*$. It follows that and $\Psi = \theta_2 \circ \Phi$. As composites of completely positive (completely bounded) maps are completely positive (completely bounded) we see that ψ_i, $i = 1, 2$ are completely bounded and Ψ is completely positive. θ is unital and so $\psi_i(\mathbf{1}) = \theta(\varphi(\mathbf{1})) = \mathbf{1}$. Together with,

$$\|\psi_i\|_{cb} = \|\theta \circ \varphi_i\|_{cb} \leq \|\theta\|_{cb} \|\varphi_i\|_{cb} \leq 1, \tag{25}$$

this proves $\|\psi_i\| = 1 = \|\psi\|_{cb}$.

Theorem 8.5 Let $\mathcal{A}$ be a C^*-algebra with unit, and let $\varphi : \mathcal{A} \to \mathcal{B}$ be completely bounded. then there exists a completely positive map $\psi : \mathcal{A} \to \mathcal{B}$ with $\|\psi\| \leq \|\varphi\|_{cb}$ such that $\psi \pm \operatorname{Re}\varphi$ and $\psi \pm \operatorname{Im}\varphi$ are all completely positive. In particular, the completely bounded maps are the linear span of the completely positive maps.

Since $\mathcal{B} \subseteq B(\mathcal{H})$ construct as in Theorem 8.5 a map $\rho : \mathcal{A} \to B(\mathcal{H})$ such that $\|\rho\|_{cb} \leq \|\varphi\|_{cb}$ and set $\psi = \theta \circ \rho : \mathcal{A} \to \mathcal{B}$. We have

$$\|\psi\|_{cb} \leq \|\theta\|_{cb} \|\rho\|_{cb} \leq \|\varphi\|_{cb}. \tag{26}$$

As θ is a projection onto $\mathcal{B}$, $\theta \circ \varphi = \varphi$. Notice that,

$$(\theta \circ \varphi^*)(a) = \theta(\varphi(a^*)^*) = \theta(\varphi(a^*))^* = ((\theta \circ \varphi)(a^*))^* = (\theta \circ \varphi)^*(a). \tag{27}$$

Consequently,

$$\theta \circ (\rho \pm \operatorname{Im}\varphi) = (\theta \circ \rho) \pm \operatorname{Im}(\theta \circ \varphi) = \psi \pm \operatorname{Im}\varphi, \tag{28}$$

is completely positive. Similarly, $\psi \pm \operatorname{Re}\varphi$ is completely positive.

8.6 Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be C^*-algebras with unit, with $\mathcal{C}$ contained in both $\mathcal{A}$ and $\mathcal{B}$, and $\mathbf{1}_{\mathcal{A}} = \mathbf{1}_{\mathcal{B}} = \mathbf{1}_{\mathcal{C}}$. Let $\mathcal{M} \subseteq \mathcal{A}$ be a subspace such that $c_1 \mathcal{M} c_2 \subseteq \mathcal{M}$ for all c_1, c_2 in $\mathcal{C}$, and set

$$\mathcal{S} = \left\{ \begin{bmatrix} c_1 & a \\ b^* & c_2 \end{bmatrix} : a, b \in \mathcal{M}, c_1, c_2 \in \mathcal{C} \right\}.$$

(i) Prove that if $\varphi : \mathcal{M} \to \mathcal{B}$ is a completely contractive $\mathcal{C}$-bimodule map, then $\Phi : \mathcal{S} \to M_2(\mathcal{B})$ defined by

$$\Phi\left(\begin{bmatrix} c_1 & a \\ b^* & c_2 \end{bmatrix} \right) = \begin{bmatrix} c_1 & \varphi(a) \\ \varphi(b)^* & c_2 \end{bmatrix},$$

is completely positive.

(ii) Prove that if $\mathcal{B}$ is injective, then the conclusions of Theorems 8.2, 8.3 and 8.5 still hold with the additonal assumption that the maps be $\mathcal{C}$-bimodule maps.

Solution:

(i) Let

$$(S_{i,j}) = \begin{bmatrix} c_{i,j}^{(1)} & a_{i,j} \\ b_{i,j}^* & c_{i,j}^{(2)} \end{bmatrix} \in M_n(\mathcal{S}). \tag{29}$$

After a canonical shuffle $(S_{i,j})$ is the matrix

$$\begin{bmatrix} H & A \\ B^* & K \end{bmatrix}, \tag{30}$$

where $H = (c_{i,j}^{(1)})$, $K = (c_{i,j}^{(2)})$, $A = (b_{i,j})$ and $B = (b_{j,i})$.Similarly after a canonical shuffle $\Phi_n((S_{i,j}))$ is

$$\begin{bmatrix} H & \varphi_n(A) \\ \varphi_n(B)^* & K \end{bmatrix}. \tag{31}$$

If the matrix in (30) is positive, then $A = B$ and H, K are positive. Let $\varepsilon > 0$ and note that the matrices $H_\varepsilon = H + \varepsilon I$ and $K_\varepsilon = K + \varepsilon I$ are positive and invertible. We have,

$$\begin{bmatrix} H_\varepsilon^{-1/2} & 0 \\ 0 & K_\varepsilon^{-1/2} \end{bmatrix} \begin{bmatrix} H_\varepsilon & A \\ A^* & K_\varepsilon \end{bmatrix} \begin{bmatrix} H_\varepsilon^{-1/2} & 0 \\ 0 & K_\varepsilon^{-1/2} \end{bmatrix} = \begin{bmatrix} I & H_\varepsilon^{-1/2} A K_\varepsilon^{-1/2} \\ K_\varepsilon^{-1/2} A^* H_\varepsilon^{-1/2} & I \end{bmatrix}. \tag{32}$$

The matrix on the left of (32) is positive and by Lemma 3.1 we have that

$$\left\| H_\varepsilon^{-1/2} A K_\varepsilon^{-1/2} \right\| \leq 1. \tag{33}$$

As φ is $\mathcal{C}$-bimodule, $\varphi_n(H_\varepsilon^{-1/2} A K_\varepsilon^{-1/2}) = H_\varepsilon^{-1/2} \varphi_n(A) K_\varepsilon^{-1/2}$. A similar calculation to (32) shows that $\Phi_n((S_{i,j}))$ is positive if and only if

$$\begin{bmatrix} I & H_\varepsilon^{-1/2} \varphi_n(A) K_\varepsilon^{-1/2} \\ K_\varepsilon^{-1/2} \varphi_n(A)^* H_\varepsilon^{-1/2} & I \end{bmatrix} \geq 0. \tag{34}$$

From the assumption that φ is completely contractive we get

$$\left\| H_\varepsilon^{-1/2} \varphi_n(A) K_\varepsilon^{-1/2} \right\| = \left\| \varphi_n(H_\varepsilon^{-1/2} A K_\varepsilon^{-1/2}) \right\| \leq 1, \tag{35}$$

which is equivalent to the matrix in (34) being positive.

(ii) We check that the maps considered in the proofs of Theorems 8.2, 8.3, and 8.5 are $\mathcal{C}$-bimodule.

Theorem 8.2 Recall that ψ was defined by

$$\Psi\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} * & \psi(a) \\ * & * \end{bmatrix}, \tag{36}$$

where Ψ is the extension of Φ described in part i). Being an extension of Φ, Ψ fixes $\mathcal{C} \oplus \mathcal{C}$, and so by Exercise 4.3, Ψ is $\mathcal{C} \oplus \mathcal{C}$-bimodule. If $c_1, c_2 \in \mathcal{C}$, then

$$\Psi\left(\begin{bmatrix} 0 & c_1 a c_2 \\ 0 & 0 \end{bmatrix} \right) = \Psi\left(\begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & c_2 \end{bmatrix} \right) \tag{37}$$

$$= \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} \Psi\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 0 & 0 \\ 0 & c_2 \end{bmatrix}, \tag{38}$$

from which we get $\psi(c_1 a c_2) = c_1 \psi(a) c_2$.

Theorem 8.3 A matrix factoring similar to the one used above shows that φ_i, for $i = 1, 2$, is a $\mathcal{C}$-bimodule map.

Theorem 8.5 We will prove that φ^* is $\mathcal{C}$-bimodule. The claims of theorem 8.5 will then follow as before.

$$\varphi^*(c_1 a c_2) = \varphi((c_1 a c_2)^*)^* = \varphi(c_2^* a^* c_1^*)^* \tag{39}$$

$$= (c_2^* \varphi(a^*) c_1^*)^* = c_1 \varphi(a^*)^* c_2 \tag{40}$$

$$= c_1 \varphi^*(a) c_2. \tag{41}$$

8.7 Let $A = (a_{i,j})_{i,j=1}^{\infty}$. Prove that the following are equivalent:

(i) $S_A : B(\ell^2) \to B(\ell^2)$ is positive.

(ii) $S_A : B(\ell^2) \to B(\ell^2)$ is completely positive.

(iii) There exists a Hilbert space $\mathcal{H}$ and a bounded sequence of vectors $\{x_i\}$ in $\mathcal{H}$ such that $a_{i,j} = \langle x_j, x_i \rangle$.

Solution: Let $P_n : \ell^2 \to \ell^2$ denote the projection onto the first n coordinates. Let $A_n = P_n A P_n$ which is the matrix equal to A in the $n \times n$ top left corner and 0 otherwise. Observe that $A_n \to_{\text{WOT}} A$ as $n \to \infty$. Note that $A_n * T = (P_n A P_n) * T = P_n(A * T)P_n = (A * T)_n$ for any $T \in B(\ell^2)$. By Theorem 3.7 the following three statements are equivalent

(a) S_{A_n} is positive for all $n \geq 1$.

(b) $A_n \geq 0$ for all $n \geq 1$.

(c) S_{A_n} is completely positive for all $n \geq 1$.

(i) implies (ii). If $T \geq 0$, then

$$S_{A_n}(T) = A_n * T = A_n * T_n = (A * T)_n = (S_A(T))_n \geq 0. \tag{42}$$

Therefore S_{A_n} is positive and by the observation above S_{A_n} is completely positive. Assume $T = (T_{i,j})_{i,j=1}^{m} \geq 0$ and let $x_j \in \ell^2$, $j = 1, \ldots, m$.

$$\begin{aligned}
\left\langle (S_A)_m(T_{i,j}) \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \right\rangle &= \sum_{i,j=1}^{m} \langle (A * T_{i,j}) x_j, x_i \rangle && (43)\\
&= \sum_{i,j=1}^{m} \lim_{n\to\infty} \langle (A * T_{i,j})_n x_j, x_i \rangle && (44)\\
&= \lim_{n\to\infty} \sum_{i,j=1}^{m} \langle (A * T_{i,j}) x_j, x_i \rangle && (45)\\
&= \lim_{n\to\infty} \sum_{i,j=1}^{m} \left\langle (S_{A_n})_m(T_{i,j}) \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \right\rangle \geq 0 && (46)
\end{aligned}$$

(ii) implies (iii). Let $S_A : B(\ell^2) \to B(\ell^2)$ be completely positive. Let $(\pi, \mathcal{H}, V)$ be a minimal Stinespring representation of S_A. Let $E_k = E_{1,k}$ and note that $E_i^* E_k = E_{i,j}$. For each $j \in \mathbb{N}$, define $x_k = \pi(E_k) V e_k \in \mathcal{H}$. Note that $\|x_k\| \leq \|V\|$. Now,

$$\begin{aligned}
\langle x_j, x_i \rangle &= \langle \pi(E_j) V e_j, \pi(E_i) V e_i \rangle && (47)\\
&= \langle V^* \pi(E_i)^* \pi(E_j) V e_j, e_i \rangle && (48)\\
&= \langle V^* \pi(E_{i,j}) V e_j, e_i \rangle && (49)\\
&= \langle S_A(E_{i,j}) e_j, e_i \rangle && (50)\\
&= \langle a_{i,j} E_{i,j} e_j, e_i \rangle && (51)\\
&= a_{i,j} && (52)
\end{aligned}$$

(iii) implies (i). Let $\|x_k\| \leq M$ for all $k \geq 1$, $T = (t_{i,j})_{i,j=1}^{\infty} \in B(\ell^2)$ and $h, k \in \ell^2$. It is enough to prove that $\|(S_A(T))_n\|$ is bounded independent of n (which establishes the fact that $S_A(T)$ is an element of $B(\ell^2)$) and $A_n \geq 0$. The latter claim implies, for a positive operator T, that

$$\langle S_A(T)h, h\rangle = \lim_{n\to\infty} \langle (S_A(T))_n h, h\rangle = \lim_{n\to\infty} \langle S_{A_n}(T)h, h\rangle \geq 0. \tag{53}$$

Now,

$$\begin{aligned}
|\langle (S_A(T))_n h, k\rangle| &= |\langle (A_n * T_n)h, k\rangle| && (54)\\
&= \left|\sum_{i,j=1}^{n} a_{i,j} t_{i,j} h_j \overline{k_i}\right| && (55)\\
&= \left|\sum_{i,j=1}^{n} \langle x_j, x_i\rangle\, t_{i,j} h_j \overline{k_i}\right| && (56)\\
&= \left|\left\langle (t_{i,j} I_{\mathcal{H}}) \begin{bmatrix} h_1 x_1 \\ \vdots \\ h_n x_n \end{bmatrix}, \begin{bmatrix} k_1 x_1 \\ \vdots \\ k_n x_n \end{bmatrix} \right\rangle\right| && (57)\\
&\leq \|(t_{i,j} I_{\mathcal{H}})\|_{B(\mathcal{H}^{(n)})} \left(\sum_{i=1}^{n} \|h_i x_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} \|k_i x_i\|^2\right)^{1/2} && (58)\\
&= \|T_n\| \left(\sum_{i=1}^{n} |h_i|^2 \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} |k_i|^2 \|x_i\|^2\right)^{1/2} && (59)\\
&\leq \|T\|\, M^2\, \|h\|\, \|k\|\,. && (60)
\end{aligned}$$

Therefore, $\|(S_A(T))_n\| \leq \|T\|\, M^2$.

Also,

$$\begin{aligned}
\langle A_n h, h\rangle_{\ell^2} &= \sum_{i,j=1}^{n} a_{i,j} h_j \overline{h_i} && (61)\\
&= \sum_{i,j=1}^{n} \langle x_j, x_i\rangle\, h_j \overline{h_i} && (62)\\
&= \sum_{i,j=1}^{n} \langle h_j x_j, h_i x_i\rangle && (63)\\
&= \left\langle \begin{bmatrix} h_1 x_1 \\ \vdots \\ h_n x_n \end{bmatrix}, \begin{bmatrix} h_1 x_1 \\ \vdots \\ h_n x_n \end{bmatrix} \right\rangle_{\mathcal{H}^{(n)}} \geq 0 && (64)
\end{aligned}$$